# The General Interval Power Function

## Diplomarbeit im Fach Informatik


vorgelegt von

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31. März 2011


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# Abstract

The present work develops and substantiates three differently extensive versions of a general power function: pow1, pow2 and pow3. Whilst the first only defines powers with positive exponents, the second additionally defines powers of zero with positive exponent, as well as powers with negative base and integral exponent. The third version exceeds the former ones and additionally defines powers with negative base and rational exponent that can be written as a fraction with odd denominator.

Interval extensions as well as reverse interval operations are defined and presented for all three versions.

To act as a reference, a Matlab implementation is given, which uses the well-known interval arithmetic library INTLAB.

# Chapter 1

# Introduction

> Give a digital computer a problem in arithmetic, and it will grind away methodically, tirelessly, at gigahertz speed, until ultimately it produces the wrong answer. …
>
> … An interval computation yields a pair of numbers, an upper and a lower bound, which are guaranteed to enclose the exact answer. Maybe you still don't know the truth, but at least you know how much you don't know. (Hayes 2003)

A major problem with inaccurate data and inexact arithmetic operations is the propagation of arithmetic errors. For simple calculations, engineers can carry out error calculations and give reasonable boundaries for the correct result. However, for large-scale computations or algorithms this is infeasible, or annoying and time-consuming at the best.

With interval arithmetic rounding errors propagate through all computations, and boundaries are given for each result of an arithmetic operation. Not only representational problems of numbers and rounding errors are negotiated. Interval arithmetic is also a valuable tool for various applications, e.g., global optimization, constraint programming, branch and bound algorithms. Additionally, it can be used to handle approximations of functions that are difficult to compute exactly.

In the past, the best achievements have been made with new methods, which are especially designed for interval arithmetics, e.g., the interval Newton method which can find *any* roots of a function in a given interval, in contrast to the usual Newton's method that only finds a single one.

Benefits are obvious, and in the past there have been several computational libraries that provide interval arithmetic functionality, but a common standard is missing, still being worked on by the IEEE working group P1788. The upcoming standard will cover many arithmetic operations, one of which will be the real power function $(x, y) \mapsto x^y$.

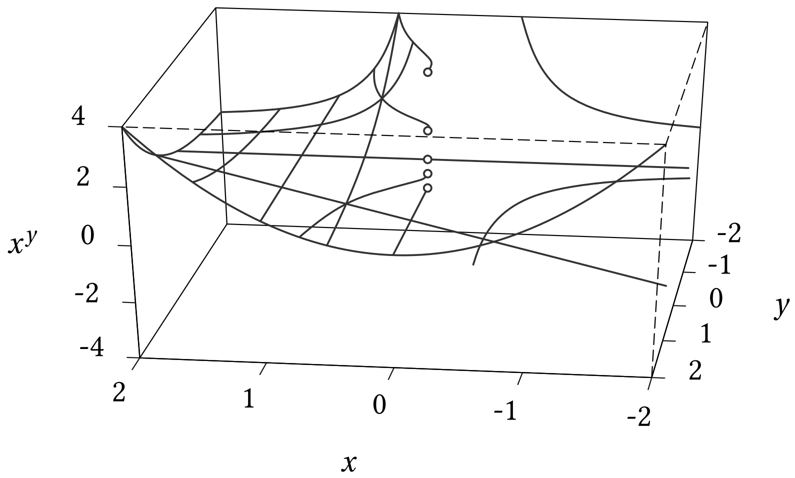


FIGURE 1.1. The general power function on real numbers is not trivial and contains several elementary functions, e.g., monomials, basic rational functions, exponential functions and root functions.

This thesis studies the possibilities of formally defining such a power function for as many pairs of real numbers as possible, making a "general-purpose" power function. Furthermore, an interval version of this general power function shall be defined as well as reverse interval operations, which are used to solve power equations, e.g., $x^2 = 4$ yields two solutions $x = \pm 2$, whereas $2^y = 4$ yields one solution: $y = 2$.

Chapter 2 highlights mathematical backgrounds of a real-valued general power function and examines several definitions of powers. Chapter 3 summarizes required parts from the current draft for the IEEE interval arithmetic standard and, in its spirit, defines general interval power functions and gives details on possible implementations. This thesis further comprises reference implementations of presented solutions, which can be found on compact disk (Appendix C overviews the implemented functionality).

# Chapter 2

# Mathematic Power Function on Real Arguments

This chapter investigates the possibility of defining a mathematically respectable power function $(x, y) \mapsto x^y$, with $x, y \in \mathbb{R}$, for use in interval arithmetic. The approach taken, as shown in Figure 2.1, starts with the definition of powers in basic algebraic structures and culminates in explaining exponentiation with unlimited real base and exponent. Along the way, attention is payed to detail, particularly with "difficult" powers resulting,

e.g., $0^0$, $(-1)^{0.5}$, and $2^{\sqrt{2}}$.

## 2.1 Powers in Semigroups and Groups

In my opinion abstract groups qualify as a very good starting point for this work: Semigroups are the most basic algebraic structure in which powers can be defined; furthermore in groups this is possible with negative exponents, plus all index laws hold. Although this explains powers with integer exponents for almost every real base in a natural way, this universal approach does not lead to fast accurate power algorithms. In any case, non-integer exponents have to be dealt with separately, which becomes clear in Section 2.2.

Wussing (2007, 15–16) describes groups as the first evolved structure in modern algebra. Due to the different roots of the group concept, which lie in "the theory of algebraic equations, number theory, and geometry" (ibid., 16), generalizations—like semigroups—have received significantly less attention. In the following only elementary concepts and few examples concerning real numbers will be needed, however. For me it is important to distinguish between powers in semigroups and powers in groups, otherwise it would be impossible to explain powers of zero at this point.

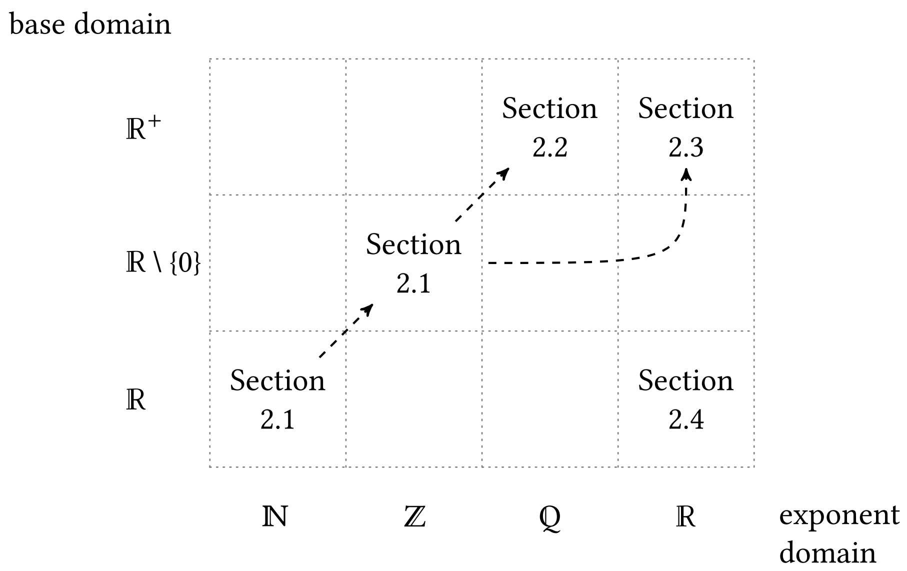


Figure 2.1. Approach taken to give a sound explanation of real exponentiation.

Definition 2.1. *"Let $\mathfrak{S}$ be a set of elements $a, b, \ldots$ and let $\circ$ be a binary operation in $\mathfrak{S}$, that is a mapping $\mathfrak{S} \times \mathfrak{S}$ into $\mathfrak{S}$ associating with every ordered pair $(a, b)$ of elements of $\mathfrak{S}$ a definite element $c$ of $\mathfrak{S}$: $c = a \circ b$. Such a set is known as a* groupoid. *... An associative groupoid is called an abstract* semi-group." (Hille and Phillips 1957, 257)

The set of real numbers together with the binary operation of multiplication, denoted $\cdot$, is the realization of prime importance for the present work. In fact this is a semigroup, because multiplication is associative in $\mathbb{R}$.

Powers have been used in abstract groups long ago, while only tacitly being "applied ... to all kinds of domains in which products are defined and where the associative law holds", which van der Waerden (1966, 16–17) substantiates to be proper. In his book Wallace (1998, 96) uses a more direct approach and points out that—owing to associativity—it is possible to "define the powers" of elements in arbitrary semigroups, i.e., in $\mathbb{R}$, "unambiguously" as follows.

Definition 2.2. *A product of $n \in \mathbb{N}$ identical factors $a \in \mathfrak{S}$ is called the $n$-th* power *of $a$ and is denoted by* $a^n = \underbrace{a \circ \cdots \circ a}_{n \text{ times}}$.

*In this notation $a$ is called the* base, *whereas $n$ is called the* exponent *of the power.*

Alternatively "this can also be stated as a 'recursive' definition $a^1 = a$, $a^{n+1} = a^n \circ a$" as remarked by Birkhoff and Mac Lane (1965, 12). However, this simple, yet important definition explains powers with positive integral exponent for every real base, and induces first index laws.

THEOREM 2.3. *Let base $a$ be an element of a semigroup with binary operator $\circ$, and let $m$ and $n$ be positive integral exponents. Then*

$$a^m \circ a^n = a^{m+n}, \tag{2.1}$$

$$(a^m)^n = a^{m \cdot n}. \tag{2.2}$$

A proof for (2.1) is given by Borůvka (1976, 148), whereas Birkhoff and Mac Lane (1965, 12) provide another variant of the proof by induction on $n$, plus a proof for (2.2) (ibid., 125).

LEMMA 2.4. *From Theorem 2.3 it follows for base $a$ and natural number exponent $n$ that*

$$a^n = \begin{cases} a & \text{if } n = 1, \\ (a \circ a)^k & \text{if } n = 2k, \text{ where } k \in \mathbb{N}, \\ (a \circ a)^k \circ a & \text{if } n = 2k+1, \text{ where } k \in \mathbb{N}, \end{cases}$$

which leads to log-time Algorithm A.1 for computation of powers in semigroups. This concludes my analysis of powers in semigroups. The concept of abstract semigroups is too limiting to allow for more effective, i.e., constant time, algorithms, or a larger domain of exponents for powers of elements. For that reason, the next step is to examine powers in groups, which extends the domain of valid exponents to the entire set of integers. Though, there is a drawback with it: The number zero is no longer allowed to act as a base.

DEFINITION 2.5. *An abstract semigroup $\mathfrak{G}$ is called an abstract* group *if the following axioms hold.*

1. *"There exists (at least) one element $e$ in $\mathfrak{G}$, called the (left)* identity, *such that $e \circ a = a$ for every element $a$ of $\mathfrak{G}$.*
2. *If $a$ is an element of $\mathfrak{G}$, there exists (at least) one element* $\text{inv}(a)$ *in $\mathfrak{G}$, called the (left)* inverse *of $a$, such that* $\text{inv}(a) \circ a = e$.*" (van der Waerden 1966, 11; the notation* $\text{inv}(a)$ *differs from the common notation $a^{-1}$ used in the original, nevertheless it is used to differentiate between the* (-1)*-st power of $a$ and the inverse of $a$ until negative powers are defined)*

DEFINITION 2.6. *Positive integral powers in groups are defined in accordance with Definition 2.2, since groups are semigroups as well. Additionally it is defined $a^0 = e$ and $a^{-n} = \text{inv}(a)^n$ for remaining integral exponents (ibid., 17).*

These definitions must not be confused with the following laws of indices. As Vinner (1998) has discovered, if one wants index laws (2.1), (2.2) to hold for arbitrary integral exponents, one "must define $a^{-n}$ to be $\text{inv}(a^n)$", finally resulting in "confusion … between definitions and theorems". A common problem that is readdressed in Subsection 2.4.1.

THEOREM 2.7. *For arbitrary group elements $a$, $b$ and integral exponents $m$, $n$ the following equations hold.*

$$a^{-1} = \mathrm{inv}(a) \tag{2.3}$$
$$(a^n)^{-1} = (a^{-1})^n \tag{2.4}$$
$$a^m \circ a^n = a^{m+n} \tag{2.5}$$
$$(a^m)^n = a^{m \cdot n} \tag{2.6}$$

PROOF. Equation (2.3) follows directly from the definition of the (-1)-st power of $a$, whereas (2.4) can be seen as a special case of (2.6). Equations (2.5), (2.6) are proven by Birkhoff and Mac Lane (1965, 125). □

Considering equations (2.5), (2.6), it is revealed that the basic index laws from Theorem 2.3 still hold in groups. Additionally, from (2.4) it becomes clear that negative powers can simply be computed by inverting the corresponding positive power—in contrast to Definition 2.6 which demands computing a power of the inverse.

The major drawback from looking at the less abstract structure of a group, especially for the intended application on real numbers, is that $\mathbb{R}$ is *not* a group with common multiplication, because there is no inverse of zero. However, $\mathbb{R} \setminus \{0\}$ is a group, and applying the above to it results in what is widely known as integral powers of real numbers. Still, powers of zero are limited to positive integral exponents at this point.

I want to conclude this section with additional index laws that do not hold in arbitrary groups, but do so in commutative groups, i.e., in real numbers, plus a brief analysis of Algorithm A.1's execution time plus a discussion of possible improvements.

LEMMA 2.8. *Let $a$, $b$ be nonzero real bases and let $m$, $n$ be integral exponents. Without loss of generality it can be assumed $m > n$. Then the following equations hold. If $n$ was positive, $a$, $b$ may actually be zero as well.*

$$a^n \cdot b^n = (a \cdot b)^n \tag{2.7}$$
$$a^n \cdot b^m = (a \cdot b)^n \cdot b^{m-n} \tag{2.8}$$
$$(-a)^n = \begin{cases} a^n & \text{if } n \text{ is even,} \\ -(a^n) & \text{otherwise} \end{cases} \tag{2.9}$$

PROOF. Multiplication is commutative in $\mathbb{R}$, thus (2.7) can easily be shown by sequentially sorting the $|2n|$ factors of $(a \cdot b)^n$. Equation (2.8) can be shown with the identity $a^n \cdot b^m = a^n \cdot b^{n+m-n} = a^n \cdot b^n \cdot b^{m-n}$ and the first equation. The latter equation is a special case of the first one with one factor being -1, which is an involution, i.e., even powers of -1 equal 1. □

In Algorithm A.1 there are two kinds of arithmetic operations. First, the exponent is being modified, i.e., either divided by 2 or decreased by 1. Both are fast operations that can be performed by bit shifting or manipulation of a single bit respectively (only odd numbers are decreased by 1). Secondly, the intermediate result and base have to be multiplied. Regarding execution time, the latter make up the most of it in various scenarios, e.g., multiplication of intervals, symmetric matrices or floating point numbers.

Therefore, the number of multiplications that the algorithm has to do needs to be minimized.

Lemma 2.9. *Let $n$ be a natural number. During computation of an $n$-th power, Algorithm A.1 performs exactly* $\mathrm{w}(n) - 1 + \lfloor \log_2 n \rfloor$ *multiplications, where* $\mathrm{w}(n)$ *denotes the Hamming weight of $n$, i.e., the sum of its binary digits.*

Proof. The correctness of the proposed formula can easily be shown by looking at the binary notation of the exponent

$$n = n_m n_{m-1} \dots n_1 n_0 = \sum_{i=0}^{m} n_i 2^i \quad (n_i \in \{0, 1\}).$$

Without loss of generality it can be stated $n_m = 1$, thus $\lfloor \log_2 n \rfloor = m$, and obviously the algorithm terminates. Until $n$ vanishes, the algorithm's while loop is executed, performing one multiplication during each cycle—unless, the exponent is odd for the first time. Hence, the number of multiplications performed is the number of cycles of the while loop minus 1, that is why it suffices to prove that the total number of cycles is $\mathrm{w}(n) + m$.

During each cycle the algorithm either divides an even exponent by 2, reducing its (binary) length by one and leaving its Hamming weight unchanged; or decreases an odd exponent by 1, decreasing its Hamming weight and leaving its length unchanged.

At the beginning of the very last cycle, the exponent has been decreased to 1, i.e., the binary length of $n$ has been reduced from $m + 1$ to 1, and its Hamming weight has been decreased by $\mathrm{w}(n) - 1$. Thus, there are $m$ plus $\mathrm{w}(n) - 1$ cycles before the last one, resulting in $m + \mathrm{w}(n)$ cycles in total. □

With the help of Lemma 2.9 it becomes clear, that the execution time of Algorithm A.1 with respect to multiplication is bounded below by $\lfloor \log_2 n \rfloor$ and bounded above by $2\lfloor \log_2 n \rfloor$, because $1 \le \mathrm{w}(n) \le \lfloor \log_2 n \rfloor + 1$.

Powers of 2 have a minimum Hamming weight of 1, thus powers with such exponents are the fastest to be computed in time $\lfloor \log_2 n \rfloor$. Hence, it is possible to decrease the number of multiplications needed for some exponents that can be

Table 2.1. Number of real $n$-th roots of a real number $x$.

| $x$ | $n$ is even | $n$ is odd |
|---|---|---|
| positive | 2 | 1 |
| zero | 1 | 1 |
| negative | 0 | 1 |

composed of powers of 2 via subtraction, e.g., $a^{2^m-1} = a^{2^m} \cdot a^{-1}$ can be computed with $m$ multiplications only ($m-1$ from $a^{2^m}$ plus another multiplication with the inverse of $a$), whereas $2^m - 1$ has a maximum Hamming weight. However, it is not trivial to find such decompositions for arbitrary numbers, and therefore I am not going to investigate this any further. A survey on such methods is given by Gordon (1998).

Similarly, from the first two equations in Lemma 2.8 it can be seen, that a product of powers can be merged, so that a power of the product of bases may be computed instead, e.g., $2^{10} \cdot 3^{10} = (2 \cdot 3)^{10}$. If exponents match, this results in computing only one power. On the other hand, if exponents do not match, there still are two powers to be computed. Whether there is a performance increase has to be determined, but can be verified with the formula from Lemma 2.9.

Summing up, it is generally advantageous to merge products of powers with equal exponents before computation of the power(s), see Equation (2.7). Other improvements may be too difficult to be feasible and worthwhile at the same time.

## 2.2 Roots and Rational Exponents

Definition 2.10. *If $a^n = x$, where exponent $n$ is a positive integer, $a$ is called an* $n$-th root *of $x$ (Spiegel 1974, 3).*

In the following I am specifically writing about powers with real bases only. This is because there is some real analysis required to show essential results. Hence, I discontinue dealing with powers in abstract algebraic structures here.

Theorem 2.11. *Let $x$ denote a real number, and let $n$ be a positive integer. Then, the exact number of real $n$-th roots of $x$ is given by Table 2.1.*

Proof. Truss (1997, 105) proves that there is a unique nonnegative real $n$-th root of $x \geq 0$. From Equation 2.9 and the simple fact, that powers of nonnegative bases are always nonnegative, it follows

1. Even powers are nonnegative. Thus, there are no $n$-th roots of $x < 0$, if $n$ is even.
2. Odd powers are nonnegative, if, and only if, the base is nonnegative. Thus, if $n$ is odd, an $n$-th root of $x \geq 0$ is nonnegative, and therefore unique.

3. If $n$ is even, for an arbitrary $n$-th root $a$ of $x \geq 0$, it follows $(-a)^n = a^n = x$. Thus, $|a|$ is the unique nonnegative $n$-th root of $x$, and $-|a|$ is another $n$-th root of $x$. Therefore, there are at most two $n$-th roots of $x$, the absolute value of which is equal.
4. An $n$-th root of $x$ is 0, if, and only if, $x = 0$, because powers of zero equal zero.
5. If $n$ is odd, $n$-th roots of $x < 0$ are negative, cf. 2. Let $a$ be an $n$-th root of $x$, then $-a$ is a positive $n$-th root of $-x$, and therefore unique. Reversely, if $a$ is the unique $n$-th root of $-x$, then $-a$ is an $n$-th root of $x$. □

DEFINITION 2.12. *Let $x$ denote a nonnegative real number. The unambiguous nonnegative $n$-th root of $x$, where $n$ is a natural number, is denoted by $\sqrt[n]{x}$. In this* radical notation *$x$ is called* radicand *and $n$ is called* exponent of the root. *Furthermore, it is defined*

$$x^{1/n} = \sqrt[n]{x}.$$

*(Hille 1964, 185; original definition extended to allow for radicand zero)*

To a certain extent, finding an $n$-th root is the inverse of exponentiation to the $n$-th power. However, Theorem 2.11 clarifies that there exists no exact inverse of exponentiation. Hence, radical notation with negative radicands—although accepted by some mathematicians, e.g., Hart (1938)—has been intentionally left undefined. That is to say, although (if $n$ is odd) there exists a real $n$-th root of negative radicands, it is not referred to via radical notation or exponential notation as in Definition 2.12. Hart (1938, 103) remarks that this would lead to contradictions when laws of indices are applied.

It makes sense to define rational powers of a negative base differently, see Subsection 2.4.2, even though laws of indices could be limited to nonnegative real bases when it comes to rational exponents.

DEFINITION 2.13. *Let $x$ denote a positive real base and let $r$ be a rational exponent. If $r = \frac{m}{n}$ with integral numerator $m$ and positive integral denominator $n$, it is defined*

$$x^r = x^{m/n} = (x^{1/n})^m.$$

*(Truss 1997, 105) If $r$ is positive, it is defined* $0^r = 0^{m/n} = (0^{1/n})^m$.

THEOREM 2.14. *Definition 2.13 agrees with integral powers defined in groups (Definition 2.6), it is well-defined, and the following index laws hold for positive real bases $x$, $y$ and rational exponents $r$, $s$.*

$$x^r \cdot x^s = x^{r+s} \tag{2.10}$$
$$(x^r)^s = x^{r \cdot s} \tag{2.11}$$
$$x^r \cdot y^r = (x \cdot y)^r \tag{2.12}$$

Proof. Any $n$-th root of zero is zero, hence $0^r = 0$ for $r \in \mathbb{Q} \setminus \{0\}$, no matter what fraction $r$ describes. For nonnegative radicand $x$ it holds $x^1 = x$, thus Definition 2.12 leads to $x = \sqrt[1]{x} = x^{1/1}$. Furthermore, for $n \in \mathbb{N}$ it follows $(x^{1/n})^n = x$, because $x^{1/n}$ is an $n$-th root of $x$. Similarly, $x$ is an $n$-th root of $x^n$, and therefore $(x^n)^{1/n} = x$. First, let $r = \frac{m}{n}$ be a whole number, i.e., $m$ is a multiple of $n$, obviously $m = n \cdot r$. Then, with Theorem 2.7 it follows $(x^{1/n})^m = (x^{1/n})^{n \cdot r} = ((x^{1/n})^n)^r = x^r$. Thus, definition of rational powers agrees with definition of integral powers.

Now, let $r = \frac{m}{n} = \frac{k \cdot m}{k \cdot n}$ with $m \in \mathbb{Z}$ and $k, n \in \mathbb{N}$, i.e., $r$ is an arbitrary rational number. Without loss of generality it suffices to prove

$$(x^{1/(k \cdot n)})^{k \cdot m} = (x^{1/n})^m$$

in order to show Definition 2.13 is well-defined. From Definition 2.12 and what has already been shown above it follows

$$\begin{aligned} x &= (x^{1/(k \cdot n)})^{k \cdot n} \\ &= ((x^{1/(k \cdot n)})^k)^n \\ \Rightarrow \quad x^{1/n} &= (x^{1/(k \cdot n)})^k \\ \Rightarrow \quad (x^{1/n})^{1/k} &= x^{1/(k \cdot n)}. \end{aligned}$$

Hence, it follows

$$\begin{aligned} (x^{1/n})^m &= (((x^{1/n})^{1/k})^k)^m \\ &= ((x^{1/(k \cdot n)})^k)^m \\ &= (x^{1/(k \cdot n)})^{k \cdot m}, \end{aligned}$$

which concludes the second part of the proof. Proving the laws of indices can easily be done by using a common denominator for the rational exponents, i.e., let

$r = \frac{m}{n}$ and $s = \frac{k}{n}$ be exponents with $m, k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
x^r \cdot x^s &= x^{m/n} \cdot x^{k/n} \\
&= (x^{1/n})^m \cdot (x^{1/n})^k \\
&= (x^{1/n})^{m+k} \\
&= x^{(m+k)/n} \\
&= x^{r+s}, \\
(x^r)^s &= (x^{m/n})^{k/n} \\
&= (((x^{1/n})^m)^{1/n})^k \\
&= (((x^{1/n^2})^{m \cdot n})^{1/n})^k \\
&= ((((x^{1/n^2})^m)^n)^{1/n})^k \\
&= ((x^{1/n^2})^m)^k \\
&= (x^{1/n^2})^{m \cdot k} \\
&= x^{(m \cdot k)/n^2} \\
&= x^{r \cdot s}, \\
x^r \cdot y^r &= x^{m/n} \cdot y^{m/n} \\
&= (x^{1/n})^m \cdot (y^{1/n})^m \\
&= (x^{1/n} \cdot y^{1/n})^m \\
&= (((x^{1/n} \cdot y^{1/n})^n)^{1/n})^m \\
&= (((x^{1/n})^n \cdot (y^{1/n})^n)^{1/n})^m \\
&= ((x \cdot y)^{1/n})^m \\
&= (x \cdot y)^{m/n} \\
&= (x \cdot y)^r.
\end{aligned}$$

□

With Theorem 2.14 it is shown, that the definition of powers with rational exponents is well-established. Additionally, as Truss (1997, 105) points out, a continuous extension could serve as a base for definition of powers with real exponents,

$$\text{e.g., } x^y = \sup\{x^r \mid r \in \mathbb{Q} \text{ and } r \le y\}$$

for all positive real bases $x$ and real exponents $y$, but "it is more elegant, however, to approach this by means of the exponential function", which is accomplished in the following section.

## 2.3 Exponential Function

Definition 2.15. *Let $x$ be a real number. Then, the* exponential function *is defined*

$$\exp x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

*(Truss 1997, 164; the notation differs from the notation in the original, in which the series starts at $n = 0$. My intention is to avoid the term $x^0$ that raises a possible problem for $x = 0$ at this point, cf. Subsection 2.4.1).*

Truss further proves that this definition is valid, i.e., the series converges for all such values of $x$, the function $\exp : \mathbb{R} \to \mathbb{R}^+$ is continuous as well as differentiable, and the following laws hold for any real $x$, $y$ and rational $r$.

$$\exp(x + y) = (\exp x) \cdot (\exp y) \tag{2.13}$$

$$\exp(r \cdot x) = (\exp x)^r \tag{2.14}$$

There exists a continuous inverse function $\log : \mathbb{R}^+ \to \mathbb{R}$ that can now be used to define powers with positive real base and real exponent. This step is motivated by the above laws and in the following proves to be valid, i.e., the following definition is the continuous extension of the power function that is mentioned at the end of the previous section.

Definition 2.16. *Let $x$ be a positive real base and let $y$ be a real exponent. Then, it is defined $x^y = \exp(y \cdot \log x)$. (Ibid., 167) If $y$ is positive, it is defined $0^y = 0$.*

This definition indicates the general power function to be continuous on the domain of positive bases and real exponents, due to the continuity of exp and log. Truss additionally shows, that the above definition equals what has already been defined in Definition 2.13 for rational exponents: For any rational exponent $r$ it follows $x^r = (\exp \log x)^r = \exp(r \cdot \log x)$, because of (2.14).

Theorem 2.17. *The following index laws hold for positive bases $x$, $x_1$, $x_2$ and real exponents $y$, $z$.*

$$x^y \cdot x^z = x^{y+z} \tag{2.15}$$

$$(x^y)^z = x^{y \cdot z} \tag{2.16}$$

$$x_1{}^y \cdot x_2{}^y = (x_1 \cdot x_2)^y \tag{2.17}$$

Proof. All equations could be proven with Theorem 2.14 and the continuity of the general power function. However, so far the definition of powers with real exponents does not depend on the definition of roots and I want to keep it this

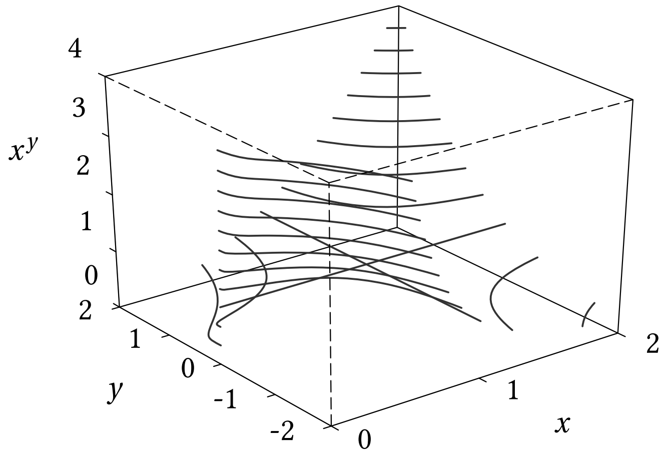


FIGURE 2.2. Contour graph of the general power function for positive bases.

way. From the definition of log and (2.13) it follows that the equation $\log(x_1 \cdot x_2) = \log x_1 + \log x_2$ is fulfilled for every positive $x_1$, $x_2$. Hence, the above equations can be proven easily.

$$
\begin{aligned}
x^y \cdot x^z &= \exp(y \cdot \log x) \cdot \exp(z \cdot \log x) \\
&= \exp(y \cdot \log x + z \cdot \log x) \\
&= \exp((y+z) \cdot \log x) \\
&= x^{y+z} \\
x^{y \cdot z} &= \exp(y \cdot z \cdot \log x) \\
&= \exp(z \cdot \log \exp(y \cdot \log x)) \\
&= (\exp(y \cdot \log x))^z \\
&= (x^y)^z \\
{x_1}^y \cdot {x_2}^y &= \exp(y \cdot \log x_1) \cdot \exp(y \cdot \log x_2) \\
&= \exp(y \cdot (\log x_1 + \log x_2)) \\
&= \exp(y \cdot \log(x_1 \cdot x_2)) \\
&= (x_1 \cdot x_2)^y
\end{aligned}
$$

□

The general power function on real arguments has certain monotonic properties on the domain of positive bases, see Figure 2.2. These properties are a result of the monotonicity of the exponential function and are used—together with its continuity—to obtain various results in Section 2.5 and Section 3.2.

THEOREM 2.18. *Let $x$, $x_1$, $x_2$ be positive real bases and let $y$, $y_1$, $y_2$ be real exponents with $x_1 < x_2$ and $y_1 < y_2$. Then, it holds*

$$x < 1 \Rightarrow x^{y_1} > x^{y_2}, \tag{2.18}$$

$$1 < x \Rightarrow x^{y_1} < x^{y_2}, \tag{2.19}$$

$$y < 0 \Rightarrow {x_1}^y > {x_2}^y, \tag{2.20}$$

$$0 < y \Rightarrow {x_1}^y < {x_2}^y. \tag{2.21}$$

PROOF. The exponential function is strictly monotonic increasing on the domain of nonnegative arguments, which can be seen from Definition 2.15. The monotonicity on negative arguments follows via the identity $\exp(-x) = (\exp x)^{-1}$ plus $\exp 0 = 1$.

For $x < 1$ it is $\log x < 0$, and thus $x^{y_1} = \exp(y_1 \cdot \log x) > \exp(y_2 \cdot \log x) = x^{y_2}$, which proves (2.18). For $1 < x$ it is $\log x > 0$, and thus (2.19) can be proven analogically.

The inverse of exp, just like the function itself, is strictly monotonic increasing, and thus $\log x_1 < \log x_2$. Hence, the order of ${x_1}^y = \exp(y \cdot \log x_1)$ and ${x_2}^y = \exp(y \cdot \log x_2)$ depends on the arithmetic sign of $y$ alone. □

LEMMA 2.19. *Let $x$ be a positive real base with $x \neq 1$. The exponential function $y \mapsto x^y$ with base $x$ has an inverse function, called* logarithm with respect to base $x$ *and denoted* $\log_x : \mathbb{R}^+ \to \mathbb{R}$.

PROOF. The inverse function is $\log_x : z \mapsto (\log z)/(\log x)$, because for $z = x^y > 0$ it holds $(\log x^y)/(\log x) = (\log \exp(y \cdot \log x))/(\log x) = (y \cdot \log x)/(\log x) = y$, if $x$ was not 1. □

## 2.4 Undefined Exponentiation

So far, the general power function is not defined on every pair of real numbers, i.e., $0^0$ is still undefined, as well as $x^y$ with negative base $x$ and non-integral exponent $y$. In many cases it can be observed that the value 1 is assigned to $0^0$, and $x^y$ is defined as $-(-x)^y$, if $y$ is rational and can be written as a fraction with an odd denominator. In the following I discuss these possibilities in detail, because they are not generally accepted in mathematics today.

### 2.4.1 0 raised to the power of 0

In the beginning of the 19th century there has been a discussion regarding the definition of $0^0$ that is continued by Knuth (1992), who sums up the historical events: While "Cauchy (1821, 70) had listed $0^0$ ... in a table of undefined forms," Libri (1833) shared the opinion of many mathematicians, that $0^0 = 1$. Thereby, he

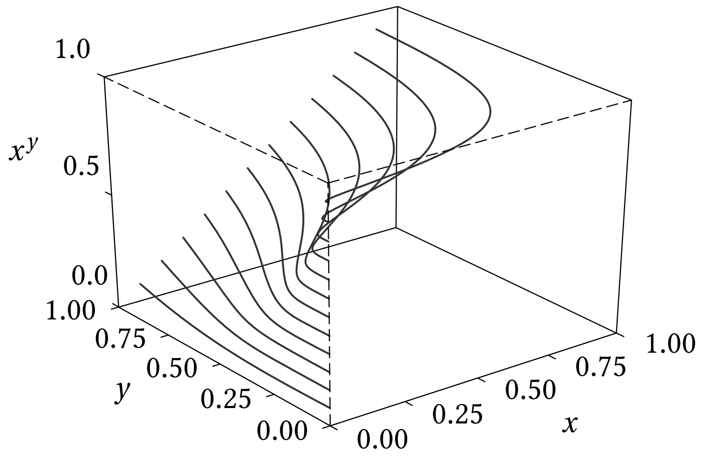


Figure 2.3. Contour graph of the general power function on the unit square.

initiated a discussion in *Journal für die reine und angewandte Mathematik*, during which Möbius (1834) proved that $x^x$ converges to 1, if $x$ is positive and converges to zero. Möbius additionally claimed that $x^y$ converges to 1, if $x$ and $y$ are positive and converge to zero, but, soon after, his claim was disproved by an anonymous author who contributed two papers (Bemerkungen 1834; Sur la valeur de $0^0$ 1834) and ended the discussion for the time being. Thus, $0^0$ was left undefined.

Similarly to Bemerkungen (1834), with $0 < c < 1$ and the sequences $(x_n)$, $(y_n)$ where $x_n = c^n$, $y_n = \frac{1}{n}$, it holds $0 < x_n, y_n \to 0$ for $n \to +\infty$ and

$$\lim_{n\to\infty} {x_n}^{y_n} = c,$$

thus, there exists no unique continuous extension of the general power function at the point $(0,0)$, as any value $c$ between 0 and 1 could be assigned to it, cf. Figure 2.3.

However, Knuth lists two reasons that may justify a definition of $0^0 = 1$. First, many formulas, e.g., the binomial theorem, can only be written in their preferred brief way, if one defines $x^0$ to equal 1 for *every* $x$. Secondly, $n^m$ equals the number of mappings in $\{1,\dots,n\}^{\{1,\dots,m\}}$ for $n, m \in \mathbb{N}$, which is correct even for either $n$ or $m$ being 0. As there exists exactly one trivial mapping in $\emptyset^\emptyset$, this is another legitimation of the definition $0^0 = 1$. Indeed, Knuth favors the opinion of Cauchy to leave $0^0$ undefined, when it describes a limiting value, while seeing trueness in Libri's point of view.

Nowadays, the value assignment $0^0 = 1$ is sometimes quietly accepted by mathematicians, e.g., for notation of a power series, or—as pointed out by Knuth—for several other formulas that the term $x^0$ appears in. Additionally, it has found its way into almost every modern programming language. While the computation of $0^0$ has to cause a "domain error" according to ANSI C from 1989 (ISO/IEC

```
In[1]:= y=x^0; z=0^x; x=0; {y,z}

     Power::indet : Indeterminate expression 0^0 encountered. >>

Out[1]= {1, Indeterminate}
```

Figure 2.4. Mathematica leaves $0^0$ undefined, but its oversimplification of the term $x^0$ allows for "computing" $0^0 = 1$. The error message belongs to the computation of the last element only.

JTC1/SC22/WG14 1998, 213), in 1996 the programming language REXX, which defines any value to the power of 0 to equal 1 (Rexx Language Association 1995, 57), has become an ANSI standard. The approach further found its way into Java (Oracle 2010), the later standard C99 of the C programming language (ISO/IEC JTC1/SC22/WG14 2007, F.10.4.4), and the IEEE-754 standard for floating point arithmetic from 2008 (IEEE 2008, 44). In a rationale for C99, the entry is justified by "significant applications where 1 is more useful than NaN" (ISO/IEC JTC1/SC22/WG14 2003, F.9.4.4), and the authors argue that "$f(t)^{g(t)}$ approaches 1 in all cases where $f$ and $g$ are analytic functions and $g(t)$ approaches zero." The same point has been made by the committee of IEEE-754, plus the argument that "such [analytic] functions are very nearly the only ones computable (in floating-point)." (Dan Zuras, pers. comm.)

It should be noted that IEEE-754 also includes a variant of the power function which is defined for non-negative bases only and refers to $0^0$ as an invalid operation—an outcome of the "controversial and not universally accepted" decision to return 1 in the standard version (ibid.).

The computational software Mathematica is one of few examples that leave $0^0$ undefined (Wolfram 2010). However, Mathematica conforms to common mathematical notation and replaces $x^0$ by 1 for output, as long as $x$ is undefined (ibid.). This behavior leads to possible pitfalls, as it can be seen in Figure 2.4. In any case, Mathematica does it correctly and interprets an exponentiation *depending on the context:* If the exponent of the power is constant zero and the base is not, the term is simplified and replaced by 1. Conversely, if the exponent is variable, there is no simplification taking place. Unfortunately, this differentiation is not an option for the general power function, which has to assign a value to an exponentiation regardless of the context.

In my opinion there is nothing wrong with assigning a value to $0^0$ as long as laws of indices are satisfied. As shown above, the general power function $(x, y) \mapsto x^y$ is discontinuous at $(0, 0)$, hence there is no point in arguing with limiting values. Defining $0^0 = 1$ is only one of two possibilities to conform to known laws of indices, because $0^0 = 0$ works just as well—complicating the notation of some formulas. Therefore, it is my point of view to leave $0^0$ undefined, as there is no *natural* reason to favor one possible value over the other. Benson (1999, 29) so aptly sums up: "The choice whether to define $0^0$ is based on convenience, not on correctness."

### 2.4.2 Negative Bases with Non-integral Exponents

Assigning a real value to a power with negative base and non-integral exponent is of a similar type to assigning a value to $0^0$. It is more useful than leaving it undefined in some applications, but, unlike a definition of $0^0$, also causes quite a few problems.

In complex analysis, there are exactly $n$ $n$-th roots of a nonzero complex number $z = r \cdot \exp(i\varphi)$, which can be written as

$$\sqrt[n]{r} \cdot \exp(i(\varphi + 2k\pi)/n). \quad (k \in \{0, \ldots, n-1\}) \tag{2.22}$$

The principal root, that is referred to by $\sqrt[n]{z}$, is obtained by choosing $k = 0$. As in Section 2.2, this definition can be used to define powers with rational—and subsequently real—exponents for every complex number base except zero. This approach is reasonable, because it conforms to roots and powers with real bases already defined above. Additionally, the consequence of which is a general power function on complex numbers that is continuous everywhere except at zero.

The principal $n$-th root of a negative real number $x$ is not a real number for $n \geq 2$, but, if $n$ is odd, the root with $k = (n+1)/2$ in Equation (2.22) happens to be the unique real $n$-th root of the negative number, cf. Theorem 2.11. The effects of denoting this non-principal root by the so far (on real numbers) undefined terms $\sqrt[n]{x}$ and $x^{1/n}$ are examined by Averbukh and Günther (2008).

Averbukh and Günther define $\mathbb{R} \to \mathbb{R}$, $x \mapsto \sqrt[n]{x}$ to be the inverse of $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^n$ for odd exponents $n$. Powers with negative bases $x$ and rational exponents $r$ can then be defined by using the irreducible fraction $\frac{m}{n} = r$, with integer $m$ and odd natural number $n$,

$$x^r = \sqrt[n]{x}^m = \sqrt[n]{x^m}.$$

Furthermore, the authors claim that $x^r = \sqrt[n]{x}^m$, if $r = \frac{m}{n}$ is *not* in irreducible form, whereas $\sqrt[n]{x^m}$ can result in a different, wrong value here. But, one might have to perform arithmetics with a complex intermediate result from the root. For example, $\sqrt[6]{(-2)^2} = \sqrt[6]{2^2} = \sqrt[3]{2} \neq (-2)^{1/3}$, but $\sqrt[6]{-2}^2 = (i\sqrt[6]{2})^2 = -\sqrt[3]{2} = (-2)^{1/3}$ works in this context.

Subsequently, Averbukh and Günther prove that there is no continuous extension of their power function on real exponents $y$, because for negative bases $x$ there exists a sequence $(y_n) \to y$ such that the sequence $(x^{y_n})$ has two different cluster points $|x|^y$ and $-|x|^y$, thus $(x, r) \mapsto x^r$ is discontinuous on $\mathbb{R}^- \times \mathbb{R}$, while being defined for the dense subset of rational exponents that can be denoted by irreducible fractions with odd denominators.

Whereas many laws of indices still are valid for the obtained general power function, Averbukh and Günther show that the index law

$$(x^r)^s = x^{r \cdot s}$$

for negative bases $x$ and rational exponents $r$, $s$ holds under very strict pre-conditions only: $(x^r)^s$ has to be defined and $s$ must equal a fraction with odd denominator.

*Why should someone accept all these disadvantages over the initially mentioned complex general power function?* According to Averbukh and Günther this kind of exponentiation is useful for some applications, including differentiation, integration and solving differential equations. They show that known laws for differentiation and integration of power functions, i.e.,

$$\frac{d}{dx} x^r = r \cdot x^{r-1},$$
$$\int x^r \, dx = (r+1)^{-1} \cdot x^{r+1} + C, \quad (r \neq -1)$$

still hold, if the exponent $r$ is rational, and $x^r$ is defined for every real number. Here, it is remarkable that their proof does not depend on a case-by-case analysis of positive and negative bases respectively, unlike the well-known approach for functions $x \mapsto \sqrt[n]{x^m}$. The avoidance of a proof by exhaustion is particularly useful for solving differential equations, where it usually has to be discriminated between many cases. (Ibid.)

---

Altogether, there are three kinds of power functions that differ in their definition of powers with negative bases: Firstly, the most limited one, i.e., limited to the real numbers, that defines such powers for integral exponents only; subsequently, a complex version that assigns complex values to powers with non-integral exponents, and is continuous as well as defined everywhere; thirdly, a sparsely defined, discontinuous function which makes use of non-principal roots, see Table 2.2. In the field of computer arithmetics all versions can be encountered. For example, the limited version is part of the IEEE standard for floating-point arithmetics (IEEE 2008), part of standard C since 1989 (ISO/IEC JTC1/SC22/WG14 1998, 2007), and applies to Java too (Oracle 2010).

The reasons for this are obvious: the result of the power function needs to be a floating-point value, thus categorically cannot represent a complex number. Additionally, an implementation of the version of the power function must be aware of the exact fractional representation of the exponent, in order to determine whether the power is defined, and whether the result is positive or negative. But, most fractions can not be described with floating-point numbers. Thus, there remains only one usable alternative for floating-point arithmetic on real numbers. Things

TABLE 2.2. Possible versions of a general real power function.

| Name | Domain | Computation |
|---|---|---|
| | restricted versions | |
| Positive Version pow1 | $\mathbb{R}^+ \times \mathbb{R}$ | $(x, y) \mapsto \exp(y \cdot \log x)$ |
| Limited Version pow2 | $\mathbb{R}^+ \times \mathbb{R}$ | $(x, y) \mapsto \exp(y \cdot \log x)$ |
| | $\{0\} \times \mathbb{R}^+$ | $(x, y) \mapsto 0$ |
| | $\mathbb{R}^- \times \mathbb{Z}$ | $(x, y) \mapsto \begin{cases} \exp(y \cdot \log \lvert x \rvert) & \text{if } y \text{ even} \\ \text{-}\exp(y \cdot \log \lvert x \rvert) & \text{if } y \text{ odd} \end{cases}$ |
| | extended versions | |
| Complex Version | $\mathbb{R}^* \times \mathbb{R}$ | $(x, y) \mapsto \exp(y \cdot \log x)$ with initial branch of complex logarithm and complex exponential function |
| | $\{0\} \times \mathbb{R}^+$ | $(x, y) \mapsto 0$ |
| Alternative Version pow3 | $\mathbb{R}^+ \times \mathbb{R}$ | $(x, y) \mapsto \exp(y \cdot \log x)$ |
| | $\{0\} \times \mathbb{R}^+$ | $(x, y) \mapsto 0$ |
| | $\mathbb{R}^- \times D$ | $(x, \frac{m}{n}) \mapsto \begin{cases} \exp(\frac{m}{n} \cdot \log \lvert x \rvert) & \text{if } m \text{ even and } n \text{ odd} \\ \text{-}\exp(\frac{m}{n} \cdot \log \lvert x \rvert) & \text{if } m \text{ odd and } n \text{ odd} \end{cases}$ with $D = \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ n \text{ odd}\}$ |

*Note:* All extended versions produce exactly the same results on the domain of the limited version, whereas the latter two versions are different extensions of the first. In particular, it holds pow1 $\subset$ pow2 $\subset$ pow3.

look different with interval arithmetics, where the other non-complex variant becomes available, which has been described and examined by Krämer and Wolff von Gudenberg (2003). I am going to deal with this possibility below in Section 3.2.

However, scientific calculators are capable of converting floating-point numbers into fractions, which generally is an approximative operation. So, calculators typically implement the last version of the power function—even though they might be capable of assessing and displaying complex number results. Texas Instruments (2008) admits that "this feature ... seems to cause some confusion," but finds "it to have practical utility." Contrariwise, computational software for computers, which also works on complex numbers, e.g., Mathematica and Matlab, does it right and implements the complex version of the power function (MathWorks 2010; Wolfram 2010).

As I am only interested in a real version of the power function, I have to dismiss the mathematically well-founded complex results for powers with negative base and non-integral exponent. Having said that, I do not recommend using the third version as described above, which generally is neither correct mathematically, nor of any significant use.

## 2.5 Limits of the power function

Lemma 2.20. Bernoulli's inequality

$$(1+x)^n \geq 1+nx$$

*is true for natural numbers $n$ and real numbers $x$ with $x \geq -1$.*

Proof. The proof of Bernoulli's inequality can be done by induction on $n$. For $n=1$ both sides of the inequality are equal, and the statement holds. Now, assume the inequality holds for some unspecified value of $n$. From the precondition $x \geq -1$ it follows $1+x \geq 0$ and both sides of the inequality may be multiplied with $1+x$, without affecting the inequality state. Thus, $(1+x)^{n+1} \geq 1+(n+1)x+nx^2$ is true, the last summand of which is nonnegative. Hence, $(1+x)^{n+1} \geq 1+(n+1)x$, which concludes the inductive step. □

From the proof it can be seen that the inequality was strict for $n>1$ and $x \neq 0$.

Corollary. *Let $x$ be a real base and let $y$ be a real exponent. Bernoulli's inequal-*

*ity induces the following basic limits.*

$$1 < x \Rightarrow \lim_{n\to\infty} x^n = +\infty \tag{2.23}$$

$$0 < x < 1 \Rightarrow \lim_{n\to\infty} x^n = 0^+ \tag{2.24}$$

$$0 < x \Rightarrow \lim_{n\to\infty} x^{1/n} = 1 \tag{2.25}$$

$$0 < y \Rightarrow \lim_{n\to\infty} n^y = +\infty \tag{2.26}$$

Proof. For the proof of (2.23), define $\varepsilon = x - 1 > 0$. Thus, Bernoulli's inequality can be applied and it follows $x^n = (1+\varepsilon)^n \geq 1 + n\varepsilon \to +\infty$. Now, if $x$ is less than 1 but greater than 0, the reciprocal of $x$ is greater than 1 and, with what has already been shown, it follows $x^n = 1/(1/x)^n \to 0^+$.

Without loss of generality it can be assumed $1 < x$ for the proof of (2.25) (the case $x = 1$ is trivial, and for $x < 1$ the limit is the reciprocal of the limit of $(1/x^{1/n})$). Obviously $(x^{1/n})$ is monotonically decreasing, cf. (2.19), and $x^{1/n} > 1$ for all $n$. Thus, $(x^{1/n})$ converges, let $z = \inf\{x^{1/n} \mid n \in \mathbb{N}\} \geq 1$ denote its limit. Assume $1 < z$. Because of (2.23), there exists $n_0 \in \mathbb{N}$ with $x < z^{n_0}$, hence $x^{1/n_0} < z$, in contradiction to the definition of $z$. Thus, 1 is the limit of $(x^{1/n})$.

Last, the proof of (2.26). There exists $m_0 \in \mathbb{N}$ with $\frac{1}{m_0} < y$. From (2.19) it follows $n^{1/m_0} \leq n^y$ for all $n$. Because of (2.21) the sequence $(n^{1/m_0})$ is monotonically increasing and it suffices to look at the subsequence $((n^{m_0})^{1/m_0}) = (n)$, which tends to $+\infty$. □

Lemma 2.21. *All relevant limiting values of the general power function are shown in Table 2.3. Each entry of the table represents the set of all possible limiting values, that can actually occur in the respective case. Wherever applicable, it has been denoted whether the values of a corresponding sequence approach the limit from above or below by notations $0^+$, $0^-$ and such.*

Proof. Because of the identities $x^y = (x^{-1})^{-y}$, $(x^y)^{-1} = (x^{-1})^y = x^{-y}$, and $|x|^y = \exp(y \cdot \log|x|) = (\exp y)^{\log|x|}$ for $x \neq 0$, cf. Theorem 2.7, Theorem 2.14 and Theorem 2.17 respectively, it suffices to prove only twenty-three entries of Table 2.3, which imply the remaining cases. In particular, the lower and the upper half of the table each possesses central symmetry, and the lower half of the table additionally is symmetric with respect to its main diagonal.

1. Case $x \to -\infty$, $y \to -\infty$. Let $(x_n)$, $(y_n)$ be sequences that tend to $-\infty$. Without loss of generality it can be assumed that $(y_n)$ is an integral sequence (otherwise ${x_n}^{y_n}$ was undefined for some $n$).
   There exists $n_0 \in \mathbb{N}$ with $x_n \leq -1$ and $y_n \leq -1$ for all $n \geq n_0$. Hence,

$$|{x_n}^{y_n}| = |x_n|^{y_n} = \left(\frac{1}{|x_n|}\right)^{-y_n} \leq \frac{1}{|x_n|} \to 0. \quad (n \geq n_0)$$

TABLE 2.3. Real limits of $x^y$.

| | $y \to -\infty$ | $y < 0$ | $y \to 0^-$ | $y = 0$ | $y \to 0^+$ | $0 < y$ | $y \to +\infty$ |
|---|---|---|---|---|---|---|---|
| | negative base | | | | | | |
| $x \to -\infty$ | $\{0^*, 0^{-\dagger}, 0^{+\ddagger}\}$ | $\{0^{-\dagger}, 0^{+\ddagger}\}$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\{-\infty^\dagger, +\infty^\ddagger\}$ | $\{-\infty^\dagger, +\infty^\ddagger\}$ |
| $x < -1$ | $\{0^*, 0^{-\dagger}, 0^{+\ddagger}\}$ | … | $\varnothing$ | … | $\varnothing$ | … | $\{-\infty^\dagger, +\infty^\ddagger\}$ |
| $x \to -1^-$ | $[-1, 1]^*$ | $\{-1^{+\dagger}, 1^{-\ddagger}\}$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\{-1^{-\dagger}, 1^{+\ddagger}\}$ | $\overline{\mathbb{R}} \setminus ]-1, 1[^*$ |
| $x = -1$ | $\{-1^\dagger, 1^\ddagger\}$ | … | $\varnothing$ | … | $\varnothing$ | … | $\{-1^\dagger, 1^\ddagger\}$ |
| $x \to -1^+$ | $\overline{\mathbb{R}} \setminus ]-1, 1[^*$ | $\{-1^{-\dagger}, 1^{+\ddagger}\}$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\{-1^{+\dagger}, 1^{-\ddagger}\}$ | $[-1, 1]^*$ |
| $-1 < x < 0$ | $\{-\infty^\dagger, +\infty^\ddagger\}$ | … | $\varnothing$ | … | $\varnothing$ | … | $\{0^*, 0^{-\dagger}, 0^{+\ddagger}\}$ |
| $x \to 0^-$ | $\{-\infty^\dagger, +\infty^\ddagger\}$ | $\{-\infty^\dagger, +\infty^\ddagger\}$ | $\varnothing$ | $\{1\}$ | $\varnothing$ | $\{0^{-\dagger}, 0^{+\ddagger}\}$ | $\{0^*, 0^{-\dagger}, 0^{+\ddagger}\}$ |
| | zero base | | | | | | |
| $x = 0$ | $\varnothing$ | … | $\varnothing$ | … | $\{0\}$ | … | $\{0\}$ |
| | positive base | | | | | | |
| $x \to 0^+$ | $\{+\infty\}$ | $\{+\infty\}$ | $\{+\infty\} \cup [1, +\infty[$ | $\{1\}$ | $[0, 1]$ | $\{0^+\}$ | $\{0^+\}$ |
| $0 < x < 1$ | $\{+\infty\}$ | … | $\{1^+\}$ | … | $\{1^-\}$ | … | $\{0^+\}$ |
| $x \to 1^-$ | $\{+\infty\} \cup [1, +\infty[$ | $\{1^+\}$ | $\{1^+\}$ | $\{1\}$ | $\{1^-\}$ | $\{1^-\}$ | $[0, 1]$ |
| $x = 1$ | $\{1\}$ | … | $\{1\}$ | … | $\{1\}$ | … | $\{1\}$ |
| $x \to 1^+$ | $[0, 1]$ | $\{1^-\}$ | $\{1^-\}$ | $\{1\}$ | $\{1^+\}$ | $\{1^+\}$ | $\{+\infty\} \cup [1, +\infty[$ |
| $1 < x$ | $\{0^+\}$ | … | $\{1^-\}$ | … | $\{1^+\}$ | … | $\{+\infty\}$ |
| $x \to +\infty$ | $\{0^+\}$ | $\{0^+\}$ | $[0, 1]$ | $\{1\}$ | $\{+\infty\} \cup [1, +\infty[$ | $\{+\infty\}$ | $\{+\infty\}$ |

*if $y$ is integral
†if $y$ is odd
‡if $y$ is even

Thus, $({x_n}^{y_n})$ is a zero sequence. If $(y_n)$ was an odd/even sequence, because of Lemma 2.8, ${x_n}^{y_n}$ would be negative/positive for $n \geq n_0$.

2. Case $x \to$ -∞, $y < 0$. Let $(x_n)$ be a sequence that tends to -∞. There exists $n_0 \in \mathbb{N}$ with $x_n \leq -1$ for all $n \geq n_0$. Assume $y \in \mathbb{Z}$, otherwise ${x_n}^y$ would be undefined for $n \geq n_0$. Obviously $y \leq -1$. Hence,

$$|{x_n}^y| = |x_n|^y = \left(\frac{1}{|x_n|}\right)^{-y} \leq \frac{1}{|x_n|} \to 0. \quad (n \geq n_0)$$

Thus, $({x_n}^y)$ is a zero sequence. Exponent $y$ is either odd or even. Because of Lemma 2.8, ${x_n}^y$ is either negative or positive for $n \geq n_0$.

3. Case $x \to$ -∞, $y \to 0^-$. Let $(x_n)$, $(y_n)$ be sequences that tend to -∞ and converge to $0^-$ respectively. There is no real limit of the sequence $({x_n}^{y_n})$, because there exists $n_0 \in \mathbb{N}$ with $x_n < 0$ and $-1 < y_n < 0$ for all $n \geq n_0$. Thus, $y_n \notin \mathbb{Z}$ and ${x_n}^{y_n}$ is undefined for $n \geq n_0$.
4. Case $x \to$ -∞, $y = 0$. Let $(x_n)$ be a sequence that tends to -∞. There exists $n_0 \in \mathbb{N}$ with $x_n < 0$ for all $n \geq n_0$. It follows ${x_n}^y = {x_n}^0 = 1$ for all $n \geq n_0$, because $x_n \neq 0$. Thus, 1 is the limit of the sequence $({x_n}^y)$.
5. Case $x < -1$, $y \to$ -∞. Let $(y_n)$ be a sequence that tends to -∞. Without loss of generality it can be assumed that $(y_n)$ is an integral sequence (otherwise $x^{y_n}$ was undefined for some $n$). Hence,

$$|x^{y_n}| = |x|^{y_n} = \left(\frac{1}{|x|}\right)^{-y_n} \to 0,$$

because of (2.24). Thus, $(x^{y_n})$ is a zero sequence. If $(y_n)$ was an odd/even sequence, because of Lemma 2.8, $x^{y_n}$ would be negative/positive.

6. Case $x < -1$, $y \to 0^-$. There is no real limit, cf. 3.
7. Case $x \to -1^-$, $y < 0$. Let $(x_n)$ be a sequence that converges to $-1^-$. Assume $y \in \mathbb{Z}$, otherwise ${x_n}^y$ would be undefined. Obviously $y \leq -1$. Hence, $({x_n}^y)$ is the product of $-y$ sequences that converge to $-1^-$, and thus converges to $(-1)^{-y}$, whereas $|{x_n}^y| < 1$ for all $n$.
8. Case $x \to -1^-$, $y \to 0^-$. There is no real limit, cf. 3.
9. Case $x \to -1^-$, $y = 0$. Let $(x_n)$ be a sequence that converges to $-1^-$. It follows ${x_n}^y = {x_n}^0 = 1$ for all $n$, because $x_n \neq 0$. Thus, 1 is the limit of the sequence $({x_n}^y)$.
10. Case $x \to -1^-$, $y \to +\infty$. Let $(x_n)$, $(y_n)$ be sequences that converge to $-1^-$ and tend to +∞ respectively. Without loss of generality it can be assumed that $(y_n)$ is an integral sequence (otherwise ${x_n}^{y_n}$ was undefined for some $n$). There exists $n_0 \in \mathbb{N}$ with $1 < y_n$ for all $n \geq n_0$. Hence, $|{x_n}^{y_n}| = |x_n|^{y_n} > |x_n| > 1$ for all $n \geq n_0$. Thus, possible limits of $({x_n}^{y_n})$ are not contained in $]-1, 1[$ in this case.
    Now, let $a$ be a real number with $1 < a$, and let $(x_n)$, $(y_n)$ be sequences with $x_n = -(a^{1/n})$, $y_n = n$ for all $n$. Then, $(x_n)$ converges to $-1^-$, cf. (2.25), $(y_n)$ tends

to $+\infty$, and it holds

$$ {x_n}^{y_n} = \begin{cases} a & \text{if } n \text{ is even,} \\ \text{-}a & \text{otherwise.} \end{cases} $$

Hence, there are subsequences of $({x_n}^{y_n})$ that converge to $a$ and $\text{-}a$ respectively. The possible limits $\pm 1$ are obtained by defining $x_n = \text{-}(2^{1/n^2})$, $y_n = n$ for all $n$, which implies

$$ {x_n}^{y_n} = \begin{cases} 2^{1/n} \to 1 & \text{if } n \text{ is even,} \\ \text{-}(2^{1/n}) \to \text{-}1 & \text{otherwise.} \end{cases} $$

In order to gain infinite limits $\pm\infty$, it can be chosen $x_n = \text{-}(2^{1/n})$, $y_n = n^2$ for all $n$. Then,

$$ {x_n}^{y_n} = \begin{cases} 2^n \to +\infty & \text{if } n \text{ is even,} \\ \text{-}(2^n) \to \text{-}\infty & \text{otherwise.} \end{cases} $$

11. Case $x = \text{-}1$, $y \to \text{-}\infty$. Let $(y_n)$ be a sequence that tends to $\text{-}\infty$. Without loss of generality it can be assumed that $(y_n)$ is an integral sequence (otherwise ${x_n}^{y_n}$ was undefined for some $n$). Then, $|x^{y_n}| = |x|^{y_n} = 1^{y_n} = 1$ and for odd/even $y_n$ the power $x^{y_n}$ is negative/positive. Thus, $(x^{y_n})$ converges to $\pm 1$, if, and only if, the exponent $y_n$ is odd/even for almost every $n$.
12. Case $x = \text{-}1$, $y \to 0^-$. There is no real limit, cf. 3.
13. Case $x = 0$, $y \to \text{-}\infty$. Let $(y_n)$ be a sequence that tends to $\text{-}\infty$. There is no real limit of the sequence $(x^{y_n})$, because there exists $n_0 \in \mathbb{N}$ with $y_n < 0$ for all $n \geq n_0$. Thus, $x^{y_n} = 0^{y_n}$ is undefined for $n \geq n_0$.
14. Case $x = 0$, $y \to 0^-$. There is no limit, cf. previous case.
15. Case $x = 0$, $y \to 0^+$. Let $(y_n)$ be a sequence that tends to $0^+$. It follows $x^{y_n} = 0^{y_n} = 0$ for all $n$, because $(y_n)$ is positive. Thus, $0$ is the limit of the sequence $(x^{y_n})$.
16. Case $x = 0$, $y \to +\infty$. Same argument as in previous case.
17. Case $x \to 0^+$, $y \to \text{-}\infty$. Let $(x_n)$, $(y_n)$ be sequences that converge to $0^+$ and tend to $\text{-}\infty$ respectively. There exists $n_0 \in \mathbb{N}$ with $x_n \leq x_{n_0} < 1$ and $y_n < 0$ for all $n \geq n_0$. Hence,

$$ {x_n}^{y_n} = \left(\frac{1}{x_n}\right)^{\text{-}y_n} \geq \left(\frac{1}{x_{n_0}}\right)^{\text{-}y_n} = {x_{n_0}}^{y_n} \to +\infty, \quad (n \geq n_0) $$

because of (2.21) and (2.24). Thus, the sequence $({x_n}^{y_n})$ tends to $+\infty$.

18. Case $x \to 0^+$, $y < 0$. Let $(x_n)$ be the monotonically decreasing sequence of unit fractions that converges to $0^+$, i.e., $x_n = 1/n$ for all $n$. Hence, ${x_n}^y = n^{-y} \to +\infty$, because of (2.26). Now, let $(x_n)$ be an arbitrary sequence that converges to $0^+$. With (2.20) it follows, that $({x_n}^y)$ has the same limit as $(1/n^y)$.
19. Case $x \to 0^+$, $y \to 0^-$. Let $(x_n)$, $(y_n)$ be sequences that converge to $0^+$ and $0^-$ respectively. There exists $n_0 \in \mathbb{N}$ with $x_n < 1$ for all $n \geq n_0$. Because of (2.20), ${x_n}^{y_n} = 1/{x_n}^{-y_n} > 1^{-y_n} = 1$ for all $n \geq n_0$. Thus, possible limits of $({x_n}^{y_n})$ are not less than 1 in this case. Now, let $a$ be a real number with $1 < a$, and let $(x_n)$, $(y_n)$ be sequences with $x_n = 1/a^n$, $y_n = -1/n$ for all $n$, cf. page 15. Then, because of (2.24), $(x_n)$ converges to $0^+$, $(y_n)$ converges to $0^-$, and it holds ${x_n}^{y_n} = a$ for all $n$. Thus, $a$ is the limit of the sequence $({x_n}^{y_n})$. The possible limit 1 is obtained by defining $x_n = 1/2^n$, $y_n = -1/n^2$ for all $n$, which implies ${x_n}^{y_n} = 2^{1/n} \to 1$, because of (2.25). In order to gain the infinite limit $+\infty$, it can be chosen $x_n = 1/2^{n^2}$, $y_n = -1/n$ for all $n$. Then, ${x_n}^{y_n} = 2^n \to +\infty$.
20. Case $x \to 0^+$, $y = 0$. Let $(x_n)$ be a sequence that converges to $0^+$. It follows ${x_n}^y = x_n^0 = 1$ for all $n$, because $x_n \neq 0$. Thus, 1 is the limit of the sequence $({x_n}^y)$.
21. Case $0 < x < 1$, $y \to 0^-$. Let $(y_n)$ be the monotonically increasing sequence of negative unit fractions that converges to $0^-$, i.e., $y_n = -1/n$ for all $n$. Then, $x^{y_n} = 1/x^{1/n} > 1$, because of (2.21), and $1/x^{1/n} \to 1$, because of (2.25). Now, let $(y_n)$ be an arbitrary sequence that tends to $0^-$. With (2.20) it follows, that $(x^{y_n})$ has the same limit as $(x^{-1/n})$.
22. Case $x \to 1^-$, $y \to 0^-$. Let $(x_n)$, $(y_n)$ be sequences that converge to $1^-$ and $0^-$ respectively. There exists $n_0 \in \mathbb{N}$ with $0 < x_n$ for all $n \geq n_0$. Because of (2.21), ${x_n}^{y_n} = 1/x_n^{-y_n} > 1^{-y_n} = 1$ for all $n \geq n_0$. Thus, $1^+$ is the limit of $({x_n}^{y_n})$, because $1^0 = 1$ and the general power function is continuous on positive bases.
23. Case $x \to 1^-$, $y = 0$. Let $(x_n)$ be a sequence that converges to $1^-$. There exists $n_0 \in \mathbb{N}$ with $0 < x_n$ for all $n \geq n_0$. It follows ${x_n}^y = {x_n}^0 = 1$ for all $n \geq n_0$. Thus, 1 is the limit of the sequence $({x_n}^y)$. □

# Chapter 3

# General Power Function on Real Intervals

Possible implementations of an interval power function have already been presented several years ago (Chiriaev and Walster 1998; Krämer, Kulisch, and Lohner 1994; Krämer and Wolff von Gudenberg 2003). Although a motion has successfully passed the IEEE Interval Standard Working Group to "provide interval extensions for the most commonly used elementary functions" (Wolff von Gudenberg 2009), which comprise the power function among others, today the working group is still far from coming to an agreement about the standardization of an interval power function, particularly if one takes into consideration how long it has taken in the process of IEEE-754, regarding the floating-point context. The diversified and inconsistent use of exponentiation in mathematics basically makes it hard, if not impossible, to find a common standard for an "'all-inclusive' power function … for arbitrary intervals" (ibid.).

In this chapter several versions of the general power function for interval arithmetic are discussed and developed. Therefore, the knowledge of the general power function from Chapter 2 acts as a background and is applied to the challenges of interval arithmetic. Yet more variants would come into play by restricting the general power function to the domain of integral exponents ($\mathrm{pown} : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$, $(x, n) \mapsto x^n$), or, less commonly, by restriction to the domain of rational exponents ($\mathrm{powr} : D \to \mathbb{R}$, $(x, m, n) \mapsto x^{m/n}$ with domain $D \subseteq \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}$), but, although being very closely related and proposed as recommended functions by Wolff von Gudenberg (2009), neither is within the scope of this thesis.

## 3.1 Theory of Intervals and Interval Functions

First of all, I briefly summarize the theory of mathematical intervals and interval functions that is required. In the following I refer to the latest draft for the IEEE Standard for Interval Arithmetic edited by Pryce (2011).

Definition 3.1. *Any member of the affinely extended real number system* $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ *is called a* finite/infinite number. *"The set of* mathematical intervals,*" which occupies the central position in interval arithmetic, is "denoted* $\overline{\mathbb{IR}}$ *[and] comprises all closed intervals of real numbers*

$$\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} \mid \underline{x} \le x \le \overline{x}\},$$

*where the bounds* $\underline{x}$, $\overline{x}$ *are extended-real numbers,* $-\infty \le \underline{x} \le \overline{x} \le +\infty$*. ... The (interval)* hull *of an arbitrary subset* $\boldsymbol{s}$ *of* $\mathbb{R}$*, written* $\operatorname{hull}(\boldsymbol{s})$*, is the tightest member of* $\overline{\mathbb{IR}}$ *that contains* $\boldsymbol{s}$ *... [, i.e.,] the intersection of all sets" containing* $\boldsymbol{s}$*. (Ibid., 4.1–3)*

In particular, mathematical intervals can be empty, bounded or unbounded, but its members are always finite numbers.

Definition 3.2. *"A* point function *is a (possibly partial) multivariate real function: that is, a mapping* $f$ *from a subset* $D$ *of* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *for some integers* $n \ge 0$, $m > 0$. *... The set* $D$ *where* $f$ *is defined is its* domain, *also written* $\operatorname{dom} f$*. The* range *of* $f$ *over an arbitrary subset* $\boldsymbol{s}$ *of* $\mathbb{R}^n$ *is the set*

$$\operatorname{range}(f, \boldsymbol{s}) = \{f(x) \mid x \in \boldsymbol{s} \text{ and } x \in \operatorname{dom} f\}.$$

*... A* box *is an interval vector* $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \overline{\mathbb{IR}}^n$*. Given an* $n$*-variable point function* $f$*, an* interval extension *of* $f$ *... is a mapping* $\boldsymbol{f}$ *from boxes* $\boldsymbol{x} \in \overline{\mathbb{IR}}^n$ *to intervals such that* $\boldsymbol{f}(\boldsymbol{x}) \supseteq \operatorname{range}(f, \boldsymbol{x})$ *for any such box* $\boldsymbol{x}$*, regarded as a subset of* $\mathbb{R}^n$*. The* sharp interval extension *of* $f$ *is defined by* $\boldsymbol{f}(\boldsymbol{x}) = \operatorname{hull}(\operatorname{range}(f, \boldsymbol{x}))$*."*

*In general, "a mapping with interval inputs and outputs is called an* interval function *if it is an interval [extension] of some point function, and an* interval mapping *otherwise." (Ibid., 4.4.5)*

With the definitions given above, an interval function $\boldsymbol{f}$ virtually evaluates its related point function $f$ for every point within a given box $\boldsymbol{x}$ where $f$ is defined, and returns an interval that contains every possible value which is taken by $f$, i.e., the respective part of the function's range.

While an interval computation propagates, producing results from intermediate results, intervals are typically getting wider and wider, especially if operations can not be carried out using sharp interval functions, which generally are non-computable. But, this is not a negative characteristic! The correct, exact result is always contained in the final (overestimated) interval. However, a question that can not be answered at this point is, whether there actually exists a result or not, as the exact result might be empty, thus, being contained in any interval. Additionally, it is not clear, whether the computation on the whole is defined for all original inputs, because, during computation, every value that is not covered by the next operation's domain can simply be ignored.

A decoration system (Hayes and Neumaier 2009), which is currently being developed, targets this dilemma and adds information about the course of computation to the result. At the best, one can draw conclusions from a result's decoration regarding the quality of the result: Whether the computed function is continuous, whether intermediate results are bounded, and whether the correct result might actually be empty. It is intended that these criteria can be used to "verify the assumptions of existence, uniqueness, or nonexistence theorems." (Pryce 2011, 4.8) However, since the decoration system is very likely going to experience major revisions, this topic is not focused upon here. The continuity/definedness/boundedness of following functions on certain subsets of their domains can be identified without further difficulties.

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In practice it is neither possible to compute sharp interval extensions of general real functions nor can arbitrary mathematical intervals be described by computers, because of memory limitation. Therefore, a number format $\mathbb{F}$ is used, which is a *finite* subset of $\overline{\mathbb{R}}$ that contains $-\infty$ and $+\infty$, i.e., $\mathbb{F}$ contains a finite number of elements. In the following this is mostly meant to be a floating-point format in the sense of IEEE-754, e.g., double precision with 53bit mantissa and 11bit exponent, but it may initially refer to any such subset of the extended reals. A number format $\mathbb{F}$ can be used to represent the infimum and supremum of so-called $\mathbb{F}$-intervals, which is very common and hereinafter referred to as "inf-sup representation."

DEFINITION 3.3. *The set of* $\mathbb{F}$-intervals *is denoted* $\overline{\mathbb{IF}}$ *and comprises all mathematical intervals* $\boldsymbol{x} = [\underline{x}, \overline{x}]$ *where the bounds* $\underline{x}$, $\overline{x}$ *are members of the number format* $\mathbb{F}$, $-\infty \leq \underline{x} \leq \overline{x} \leq +\infty$. *The (interval)* $\mathbb{F}$-hull *of an arbitrary subset* $\boldsymbol{s}$ *of* $\mathbb{R}$, *written* $\text{hull}_{\mathbb{F}}(\boldsymbol{s})$, *is the tightest member of* $\overline{\mathbb{IF}}$ *that contains* $\boldsymbol{s}$.

*An* $\mathbb{F}$-interval extension *of a point function* $f$ *is a mapping* $\boldsymbol{f}_{\mathbb{F}}$ *from boxes* $\boldsymbol{x} \in \overline{\mathbb{IF}}^n$ *to* $\mathbb{F}$*-intervals that additionally is an interval extension of* $f$, *regarding* $\boldsymbol{f}_{\mathbb{F}}$ *as a mapping from* $\overline{\mathbb{IR}}^n$ *to* $\overline{\mathbb{IR}}$. *The* sharp $\mathbb{F}$-interval extension *of* $f$ *is defined by* $\boldsymbol{f}_{\mathbb{F}}(\boldsymbol{x}) = \text{hull}_{\mathbb{F}}(\text{range}(f, \boldsymbol{x}))$.

### 3.1.1 Rounding and Rounding Errors

DEFINITION 3.4. Rounding up and down, *i.e., rounding towards plus/minus in-*

*finity, are mappings from real numbers to a number format* $\mathbb{F}$ *that satisfy*

$$\triangle : \mathbb{R} \to \mathbb{F},\ x \mapsto \min\{y \in \mathbb{F} \mid x \le y\},$$
$$\triangledown : \mathbb{R} \to \mathbb{F},\ x \mapsto \max\{y \in \mathbb{F} \mid y \le x\}.$$

The result of a rounding operation may be the unbound value $+\infty$ or $-\infty$ from $\mathbb{F}$. Rounding operators play an important role in the following, because sharp $\mathbb{F}$-interval extensions of functions can be defined by rounding the respective interval boundaries of the sharp interval extension in the context of $\mathbb{F}$. For example, let $f$ be a point function. Then, the value of $f$'s sharp $\mathbb{F}$-interval extension $\boldsymbol{f}_{\mathbb{F}}$ on an $\mathbb{F}$-interval $\boldsymbol{x}$ is $[\triangledown f(a), \triangle f(b)]$, if $f$, restricted to $\boldsymbol{x}$, takes its minimum/maximum in $a$ and $b$ respectively. More generally, it holds $\text{hull}_{\mathbb{F}}([\underline{x}, \overline{x}]) = [\triangledown \underline{x}, \triangle \overline{x}]$ for arbitrary mathematical intervals with boundaries $-\infty \le \underline{x} \le \overline{x} \le +\infty$.

Rounding of interval boundaries in the respective floating-point context is responsible for above-mentioned errors—or inaccuracies in the sense of not being sharp enclosures—of results in interval arithmetics. When using floating-point arithmetics as in IEEE-754, the errors introduced by rounding can be handled easily: Upward and downward rounding within the scope of normalized numbers introduces a relative error which is bounded above by the mantissa's inaccuracy. In the context of double precision numbers it holds $\triangle x \le (1+\varepsilon)\cdot x$ with $\varepsilon = 2^{-52}$ for any real number $x$ between the lowest positive normalized number $2^{-1022}$ and the greatest finite number $(1-2^{-53})\cdot 2^{1024}$ . Contrariwise, within the scope of denormalized numbers it is more appropriate to look at the absolute error, because relative rounding errors increase for these very low numbers. The absolute error depends on both mantissa length and exponent's range. In double precision it is $\triangle x \le x + \delta$ with $\delta = 2^{-52-1022} = 2^{-1074}$ for any real number $x$ between zero and the lowest positive normalized number. Analogous conclusions can be made for negative numbers and/or downward directed rounding. Summing up, in a double precision floating-point context it holds

$$|x| \in [0, 2^{-1022}] \Rightarrow |\diamond x - x| \le \delta, \tag{3.1}$$
$$|x| \in [2^{-1022}, (1-2^{-53})\cdot 2^{1024}] \Rightarrow \left|\frac{\diamond x - x}{x}\right| \le \varepsilon, \tag{3.2}$$

with $\diamond \in \{\triangle, \triangledown\}$, $\delta = 2^{-1074}$, and $\varepsilon = 2^{-52}$, cf. Neumaier (1990, 1.3).

### 3.1.2 Overlapping Relation

In the following, a notation that describes the relative position of two intervals is very useful. Nehmeier and Wolff von Gudenberg (2010) describe an interval overlapping relation $\circledcirc$ and name thirteen states that a pair of non-empty intervals can

Table 3.1. Overlapping states of non-empty intervals.

| $[\underline{a},\overline{a}] \circledcirc [\underline{b},\overline{b}]$ | Condition |
|---|---|
| beforeP | $\underline{a} \le \overline{a} < \underline{b} \le \overline{b}$ |
| meets | $\underline{a} < \overline{a} = \underline{b} < \overline{b}$ |
| overlaps | $\underline{a} < \underline{b} < \overline{a} < \overline{b}$ |
| starts | $\underline{b} = \underline{a} \le \overline{a} < \overline{b}$ |
| containedByP | $\underline{b} < \underline{a} \le \overline{a} < \overline{b}$ |
| finishes | $\underline{b} < \underline{a} \le \overline{a} = \overline{b}$ |
| equalP | $\underline{b} = \underline{a} \le \overline{a} = \overline{b}$ |
| finishedBy | $\underline{a} < \underline{b} \le \overline{b} = \overline{a}$ |
| containsP | $\underline{a} < \underline{b} \le \overline{b} < \overline{a}$ |
| startedBy | $\underline{a} = \underline{b} \le \overline{b} < \overline{a}$ |
| overlappedBy | $\underline{b} < \underline{a} < \overline{b} < \overline{a}$ |
| metBy | $\underline{b} < \overline{b} = \underline{a} < \overline{a}$ |
| afterP | $\underline{b} \le \overline{b} < \underline{a} \le \overline{a}$ |

*Source:* Nehmeier and Wolff von Gudenberg (2010, Table 2).
*Note:* See detachable card as an appendix for examples.

take, an enumeration is given in Table 3.1. Their motion has passed the IEEE Interval Standard Working Group, and overlapping states can be efficiently evaluated (ibid.).

Additionally, a more complicated overlapping relation $\circledcirc\!\!\circ$ is needed that can resolve overlapping states in greater detail. If two intervals are compared and their intersection is not empty, it is impossible to infer from most states of $\circledcirc$ whether the midpoint of the second interval is part the intersection, or where the midpoint is located relative to the intersection—an important point which is frequently required in the following. The midpoint's position can simply be considered by evaluation of the overlapping relation $\circledcirc$ for the two halves of the second interval, i.e., $\circledcirc\!\!\circ : ([\underline{a},\overline{a}],[\underline{b},\overline{b}]) \mapsto ([\underline{a},\overline{a}] \circledcirc [\underline{b},(\underline{b}+\overline{b})/2], [\underline{a},\overline{a}] \circledcirc [(\underline{b}+\overline{b})/2,\overline{b}])$. If $[\underline{b},\overline{b}]$ is bound, and $\underline{b} < \overline{b}$—otherwise one could hardly call it an actual midpoint—, there are exactly twenty-six relevant states of $[\underline{a},\overline{a}] \circledcirc\!\!\circ [\underline{b},\overline{b}]$ that can occur, all of which are defined and listed in Table 3.2, see detachable card as an appendix for sketches of each state.

In a floating-point context the midpoint can sometimes not be computed exactly, making the evaluation more complicated. However, it does not happen in the following, where the $\circledcirc\!\!\circ$ relation is exclusively used with the second interval

TABLE 3.2. Overlapping states of non-empty intervals with consideration of the second interval's midpoint.

| | Condition | |
|---|---|---|
| $[\underline{a},\overline{a}]$⚯$[\underline{b},\overline{b}]$ | $[\underline{a},\overline{a}]$⚭$[\underline{b},m]$ | $[\underline{a},\overline{a}]$⚭$[m,\overline{b}]$ |
| Ⓐ | beforeP | beforeP |
| Ⓑ | meets | beforeP |
| Ⓒ | overlaps | beforeP |
| Ⓓ | starts | beforeP |
| Ⓔ | containedByP | beforeP |
| Ⓕ | finishes | meets |
| Ⓖ | finishes | starts |
| Ⓗ | equalP | meets |
| Ⓘ | finishedBy | meets |
| Ⓙ | containsP | overlaps |
| Ⓚ | containsP | finishedBy |
| Ⓛ | containsP | containsP |
| Ⓜ | startedBy | overlaps |
| Ⓝ | startedBy | finishedBy |
| Ⓞ | startedBy | containsP |
| Ⓟ | overlappedBy | overlaps |
| Ⓠ | overlappedBy | finishedBy |
| Ⓡ | overlappedBy | containsP |
| Ⓢ | metBy | starts |
| Ⓣ | metBy | equalP |
| Ⓤ | metBy | startedBy |
| Ⓥ | afterP | containedByP |
| Ⓦ | afterP | finishes |
| Ⓧ | afterP | overlappedBy |
| Ⓨ | afterP | metBy |
| Ⓩ | afterP | afterP |

*Note:* $m = (\underline{b}+\overline{b})/2$. It is assumed $\underline{b} < \overline{b}$, otherwise there were three more cases. Graphical sketches of respective examples are given on a detachable card as an appendix.

being [-1,1], hence I do not discuss this problem further.

### 3.1.3 Reverse Operations

There are a few difficulties with inversion of interval functions. Plain solutions may create results that are either of low quality, i.e., by far overvalue the correct answer, or are wrong. For example, in the equation $x^y = z$ for given real $y$ and $z$, positive solutions $x$ can be determined via $x = z^{1/y}$, if $y \neq 0$. Now, in an analogous equation $x^{\boldsymbol{y}} = \boldsymbol{z}$ with mathematical intervals $\boldsymbol{y}$ and $\boldsymbol{z}$, solutions $x \in \mathbb{R}$ are called valid, if, and only if, there exists $y \in \boldsymbol{y}$ such that $x^y \in \boldsymbol{z}$. However, an interval computation $\boldsymbol{z}^{1/\boldsymbol{y}}$ generally does not produce the exact set of results for this equation. On the one hand, for $\boldsymbol{y} = [-.5, .4]$ and $\boldsymbol{z} = [.25, .25]$ it is $\boldsymbol{z}^{1/\boldsymbol{y}} = \boldsymbol{z}^{]-\infty,+\infty[} = [0, \infty[$. But, the actual set of positive solutions was $]0, .03125] \cup [16, +\infty[$, cf. Table B.1. In constraint programming, when it was known that $x \in [4, 20]$, there is a problem, as an exact computation could further improve the constraints on $x$ toward $[16, 20]$.

On the other hand, for $\boldsymbol{y} = [0, 0]$ and $\boldsymbol{z} = [.5, 1]$ it is $\boldsymbol{z}^{1/\boldsymbol{y}} = \boldsymbol{z}^{\emptyset} = \emptyset$, in contrast to all positive $x$ being actual solutions. The latter problem could be solved using containment sets: Pryce and Corliss (2006) have studied the effects of loose evaluation of division by zero, which applies here, but the current draft of the standard for interval arithmetic does not follow the containment set theory approach.

Neumaier (2008) has proposed "reverse operations" to act as an effective resolution for the problems encountered: A single operation shall compute an interval containing solutions to basic equations, which comprise intervals, interval operations and optional interval constraints. While some interval operations are not required in reverse mode, e.g., plus and minus, Neumaier demands reverse power operations (ibid., Table 3.10).

A formal definition of reverse interval operations has been given by Nehmeier (2010) next to a summary of reverse multiplication and reverse division.

DEFINITION 3.5. *For a (partial) binary arithmetic operation* $\circ$ *there are two* binary reverse operations *on intervals,* $\circ_1^- : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \wp(\mathbb{R})$ *and* $\circ_2^- : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \wp(\mathbb{R})$*, defined by*

$$\circ_1^-(\boldsymbol{y}, \boldsymbol{z}) = \{x \in \mathbb{R} \mid \text{there exists } y \in \boldsymbol{y} \text{ with } x \circ y \in \boldsymbol{z}\} \text{ and}$$
$$\circ_2^-(\boldsymbol{x}, \boldsymbol{z}) = \{y \in \mathbb{R} \mid \text{there exists } x \in \boldsymbol{x} \text{ with } x \circ y \in \boldsymbol{z}\}$$

*with* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \overline{\mathbb{IR}}$ *(ibid., 2.2).*

These reverse operations play a central role, because they can be used to define reverse hull operations with constraints given to the result. For the equation $x \circ y = z$, let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \overline{\mathbb{IR}}$ be constraints on the (real) variables $x$, $y$ and $z$. Constraints may be—but need not be—very weak, e.g., $\boldsymbol{x} = ]-\infty, +\infty[$. Then, there are two operations

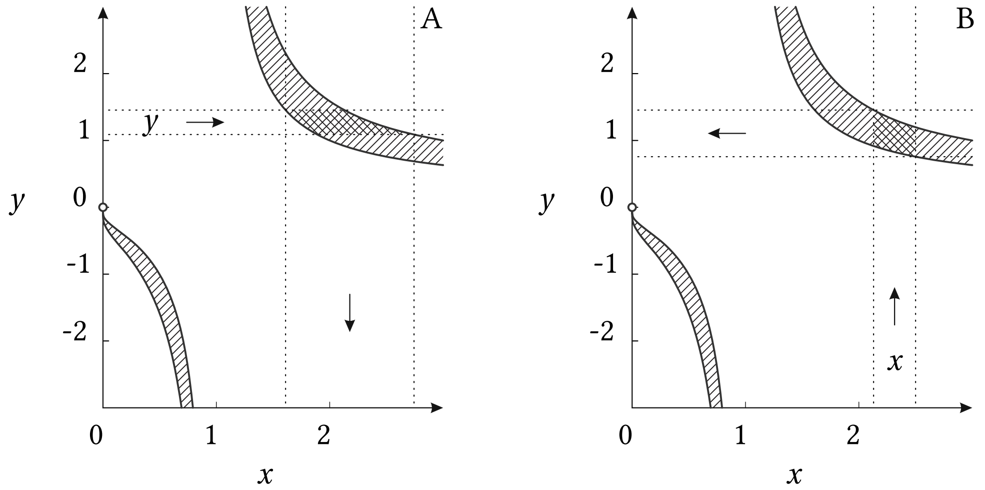


FIGURE 3.1. Graphical interpretation of reverse power operations. The (two-dimensional) inverse image of $\boldsymbol{z}$ is hatched and outlined in the figure, in the shown example it is $\boldsymbol{z} = [2,3]$. *A*, for given intervals $\boldsymbol{y}$ and $\boldsymbol{z}$ find every $x$ with $\{x\}^{\boldsymbol{y}} \cap \boldsymbol{z} \neq \emptyset$. *B*, for given intervals $\boldsymbol{x}$ and $\boldsymbol{z}$ find every $y$ with $\boldsymbol{x}^{\{y\}} \cap \boldsymbol{z} \neq \emptyset$.

$\bullet_1^-$ and $\bullet_2^-$ that can further optimize, i.e., narrow, the constraints on the variables $x$ and $y$ respectively on the domain of real intervals (a sharp interval evaluation $(\boldsymbol{x} \circ \boldsymbol{y}) \cap \boldsymbol{z}$ would naturally yield the optimization of the constraints on $z$ for any continuous operation $\circ$). It can be defined $\bullet_1^- : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \overline{\mathbb{IR}}$ and $\bullet_2^- : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \overline{\mathbb{IR}}$ with

$$\bullet_1^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \text{hull}(\{x \in \boldsymbol{x} \mid \text{there exists } y \in \boldsymbol{y} \text{ with } x \circ y \in \boldsymbol{z}\}),$$
$$\bullet_2^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \text{hull}(\{y \in \boldsymbol{y} \mid \text{there exists } x \in \boldsymbol{x} \text{ with } x \circ y \in \boldsymbol{z}\}).$$

The binary reverse operations, which yield power sets, can easily be used to define and develop the practically useful ternary reverse interval operations, because of the identities $\bullet_1^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \text{hull}(\boldsymbol{x} \cap \circ_1^-(\boldsymbol{y}, \boldsymbol{z}))$ and $\bullet_2^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \text{hull}(\boldsymbol{y} \cap \circ_1^-(\boldsymbol{x}, \boldsymbol{z}))$. (Ibid.)

Thus, it suffices to present reverse operations $\circ_1^-$ and $\circ_2^-$ for $\circ$ being the general power function below. Evaluation of the binary reverse operations for a single pair of intervals is not difficult graphically, see Figure 3.1, but yields a lot of special cases to be considered.

## 3.2 Variants of the General Interval Power Function

In Chapter 2 the intention is to explain exponentiation for as many pairs of base and exponent as possible. But, as mentioned in Section 2.4, the definition of a power for each such pair depends on the context. Strictly speaking, this problem eventually comes up at the point where reasonable doubts about the definition $0^0$ and powers with negative base in correlation with a non-integral exponent arise. There is already a problem at an earlier stage: The function pown : $(x, n) \mapsto x^n$ maps base and integral exponent pairs to their power. Does the function's domain comprise the pair $(0, n)$ for $n$ being a natural number, i.e., can this function indeed evaluate $0^n = 0$? Admittedly, the set of natural numbers is a subset of the set of integral numbers, and thus, it is possible to use a *fallback solution* and apply the definition of powers in semigroups in the case where the exponent is positive. However, does this really describe what one expects this function to do? One might want to compute powers with integral exponents within the group $\mathbb{R} \setminus \{0\}$.

Fortunately, this discrimination only makes a difference when powers with base zero are to be computed. The power functions $x \mapsto x^n$ ($n \in \mathbb{N}$) are continuous, and thus, for the strict interval extension of pown the matter is only relevant for powers with a base that equals the point interval $[0, 0] = \{0\}$.

Things look rather different in the context of powers with rational or real exponents, see definitions 2.13, 2.16. Here, the definition of powers with base zero seems somewhat artificial: In the first case, with rational exponents, the definition of powers with base zero uses a fallback solution and makes use of powers restricted to positive integral exponents, which—like with pown—can be seen as invalid. In the latter case, with real exponents, the equivalent definition can only be based on limiting values, because the expression $\exp(y \cdot \log x)$ is not defined for $x = 0$, even for complex versions of exp and log.

Furthermore, when looking at powers with non-integral exponent, one might not be interested in powers with non-positive base at all, e.g., IEEE-754 includes a function powr that is "derived by considering only $\exp(y \cdot \log x)$," and thus, is only defined for $x > 0$ whereas limits have to do in the case where $x = 0$ and $y \neq 0$ (IEEE 2008).

Hence, for practical application it seems very helpful to offer different versions of the power function. The question is: How many useful variants are there, and how are they related?

In Chapter 2 the limited version and the real extended version, pow2 and pow3 in Table 2.2 on page 19, have been defined bottom-up, cf. Figure 2.1. Though, there is another way of defining them: The complex version of the power function, which uses the initial branch of the complex logarithm, can be restricted to real numbers. When only evaluated where results are real, the complex version equals pow2. The complex version is well-established in complex analysis and offers great properties, e.g, is differentiable. This qualifies pow2 for a very good candidate for the general (real) power function.

Alternatively, one can define a multi-valued complex power function that comprises all branches of the complex logarithm. When only evaluated where on of its values is real, this multi-valued variant happens to equal pow3, cf. non-principal roots in Subsection 2.4.2.

### 3.2.1 Positive Version

On the domain of positive bases it is very clear how the general power function should be defined, cf. Definition 2.16. The first, most restricted version of the general power function pow1 acts as a starting point for the coming versions. It is defined

$$\text{pow1} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R},\ (x, y) \mapsto \exp(y \cdot \log x).$$

Apart from being well-founded, this version offers the advantage, that its (mathematical) strict interval extension can be defined using strict interval extensions of logarithm, multiplication and exponential function. In practice, when using floating-point numbers, one gets a pretty good $\mathbb{F}$-interval extension by using sharp $\mathbb{F}$-interval extensions of the elementary functions mentioned above, but this approach neither results in the sharp $\mathbb{F}$-interval extensions of pow1 as rounding errors accrete, nor does it lead to an efficient computation of the result.

The effort of computing an interval extension pow1 can be decreased by taking advantage of monotonic properties of the power function, cf. Theorem 2.18, as it has already been published by Krämer, Kulisch, and Lohner (1994). While log and exp are monotonically increasing, the interval extensions of these functions can simply be computed by evaluating the respective point functions for interval boundaries in floating-point (with directed rounding). Contrariwise, interval multiplication is complicated in general, but can be simplified when neither factor contains 0 as an inner point. Additionally, when the latter is the case, it might happen that it suffices to know only one of infimum and supremum of the logarithm of the $\mathbb{F}$-interval that represents the base, which can save some more time, because logarithms are quite difficult to compute.

Let $\boldsymbol{x} = [\underline{x}, \overline{x}]$ and $\boldsymbol{y} = [\underline{y}, \overline{y}]$ be $\mathbb{F}$-intervals and $\boldsymbol{z} = [\underline{z}, \overline{z}] = \text{pow1}_{\mathbb{F}}(\boldsymbol{x}, \boldsymbol{y})$ shall be computed, with the canonical $\mathbb{F}$-interval extension $\text{pow1}_{\mathbb{F}}$ of pow1 which makes use of logarithm, multiplication and exponential function. Considering the intermediate results, where $\triangle/\triangledown$ denote directed rounding operations in the context $\mathbb{F}$, in Algorithm 3.1 it can be seen, that, in general, there are two logarithms, eight multiplications (with directed rounding in the number format context) and two exponential functions to be computed. Discrimination between several cases for parameters $\boldsymbol{x}$ and $\boldsymbol{y}$ leads to significant improvements in efficiency on platforms which do not support parallelization of these operations. The intended improvements have not yet been entirely realized by known publications on the topic. Whilst the knowledge of the exact result and an optimized program code already

$$
\begin{aligned}
[\underline{l},\overline{l}] &= \log_{\mathbb{F}}[\underline{x},\overline{x}] \\
&= [\triangledown \log \underline{x}, \vartriangle \log \overline{x}], \\
[\underline{m},\overline{m}] &= [\underline{y},\overline{y}] \bullet_{\mathbb{F}} [\underline{l},\overline{l}] \\
&= [\min\{\triangledown(\underline{y}\cdot\underline{l}), \triangledown(\underline{y}\cdot\overline{l}), \triangledown(\overline{y}\cdot\underline{l}), \triangledown(\overline{y}\cdot\overline{l})\}, \\
&\qquad \max\{\vartriangle(\underline{y}\cdot\underline{l}), \vartriangle(\underline{y}\cdot\overline{l}), \vartriangle(\overline{y}\cdot\underline{l}), \vartriangle(\overline{y}\cdot\overline{l})\}], \\
[\underline{z},\overline{z}] &= \exp_{\mathbb{F}}[\underline{m},\overline{m}] \\
&= [\triangledown \exp \underline{m}, \vartriangle \exp \overline{m}],
\end{aligned}
$$

ALGORITHM 3.1. Naive approach to compute powers with positive base in floating-point.

TABLE 3.3. The value of pow1($[\underline{x},\overline{x}],[\underline{y},\overline{y}]$) with $0 < \underline{x}$.

| | $\overline{x} \le 1$ | $\underline{x} < 1 < \overline{x}$ | $1 \le \underline{x}$ |
|---|---|---|---|
| $0 \le \underline{y}$ | $[\underline{x}^{\overline{y}}, \overline{x}^{\underline{y}}]$ | $[\underline{x}^{\overline{y}}, \overline{x}^{\overline{y}}]$ | $[\underline{x}^{\underline{y}}, \overline{x}^{\overline{y}}]$ |
| $\underline{y} < 0 < \overline{y}$ | $[\underline{x}^{\overline{y}}, \underline{x}^{\underline{y}}]$ | $\mathrm{hull}([\underline{x}^{\overline{y}}, \underline{x}^{\underline{y}}] \cup [\overline{x}^{\underline{y}}, \overline{x}^{\overline{y}}])$ | $[\overline{x}^{\underline{y}}, \overline{x}^{\overline{y}}]$ |
| $\overline{y} \le 0$ | $[\overline{x}^{\overline{y}}, \underline{x}^{\underline{y}}]$ | $[\overline{x}^{\underline{y}}, \underline{x}^{\underline{y}}]$ | $[\overline{x}^{\underline{y}}, \underline{x}^{\overline{y}}]$ |

*Note:* Cases with $\underline{x} \le 0$ yield special-cases, but are trivial. In the second row and the first or last column, there are cases where only one boundary of the interval $[\underline{x},\overline{x}]$ takes part. In these cases it suffices to compute a single logarithm.

saves six multiplications in cases where $(1,0)$ is no inner point of $\boldsymbol{x} \times \boldsymbol{y}$, see Table 3.3, dynamic programming can be used to save the calculation of a logarithm in some cases where $\underline{y} < 0 < \overline{y}$. Algorithm A.2 comprises all mentioned optimizations and is approximately 60% faster than the naive approach when using INTLAB (Rump 1999) on a notebook with Intel Core Duo processor.

However, the algorithm still is not perfect, because rounding errors occur, and, in general, one does not obtain the $\mathbb{F}$-hull of the exact result. The major reason at this point is, that log and exp have no rational points except from $(1,0)$ and $(0,1)$ respectively. Floating-point numbers usually represent rational numbers, so rounding of results from these functions almost always induces a small error. Unfortunately, several powers of floating-point numbers actually *are rational* and could often be represented in the floating-point context. Yet, with above approach, the correct, i.e., tightest, result categorically cannot be computed. Lauter and Lefèvre

(2007) have studied rounding problems of the power function in detail, and have stated that (when using rounding to nearest number) most of the problematic cases occur when the exponent is 2, 3, 4, or 3/2, which should be treated separately. When using directed rounding, i.e., $\triangle$ and $\triangledown$, the situation is more difficult, because a lot more results meet a rounding boundary (Lauter and de Dinechin 2009). However, an estimation on the worst-case error can be given.

Lemma 3.6. *Let* $[\underline{x}, \overline{x}]$ *be a base interval and let* $[\underline{y}, \overline{y}]$ *be an exponent interval, both with boundaries in double precision, i.e.,* $\underline{x}, \overline{x}, \underline{y}, \overline{y} \in \mathbb{D}$*. Let* $[\underline{z}, \overline{z}] \in \overline{\mathbb{ID}}$ *be the result of the canonical* $\mathbb{D}$*-interval extension* $\mathrm{pow1}_{\mathbb{D}}$*, that is* $[\underline{z}, \overline{z}] = \exp_{\mathbb{D}}([\underline{y}, \overline{y}] \bullet_{\mathbb{D}} \log_{\mathbb{D}}[\underline{x}, \overline{x}])$*. Normalized, finite interval boundaries* $\widehat{z} \in \{\underline{z}, \overline{z}\}$ *have a worst-case relative error of* $\varepsilon = 2^{-41}$ *compared to the exact result.*

Proof. Let $x \in \mathbb{D}$ be a positive base and $y \in \mathbb{D}$ be a positive exponent such that both $\log x$ and $y \cdot \triangle \log x$ being at least the smallest positive normalized number of $\mathbb{D}$, i.e., they are greater than or equal to $2^{-1022}$. In particular it holds $1 < x$ and $1 < x^y$. Additionally, it is assumed that $\widehat{z} = \triangle \exp(\triangle(y \cdot \triangle \log x))$ can be computed and is less than $+\infty$.

Calculation of $\triangle x^y$ with maximum precision ensures $\triangle x^y \le (1+\varepsilon) \cdot x^y$ with $\varepsilon = 2^{-52}$ for floating-point with double precision, cf. (3.2). During the computation of $\widehat{z}$ there are three rounding operations that increase the relative error of $\widehat{z}$ compared to the correct result $x^y$.

$$\begin{aligned}
\triangle \log x &\le (1+\varepsilon) \cdot \log x \\
\Rightarrow \triangle(y \cdot \triangle \log x) &\le (1+\varepsilon) \cdot (y \cdot (1+\varepsilon) \cdot \log x) \\
&= (1+\varepsilon)^2 \cdot y \cdot \log x \\
\Rightarrow \widehat{z} = \triangle \exp(\triangle(y \cdot \triangle \log x)) &\le (1+\varepsilon) \cdot \exp((1+\varepsilon)^2 \cdot y \cdot \log x) \\
&= (1+\varepsilon) \cdot x^y \cdot (x^y)^{2\varepsilon+\varepsilon^2}
\end{aligned}$$

It is known that $x^y$ is at most the greatest finite number of $\mathbb{D}$. Hence,

$$\begin{aligned}
\widehat{z} &< (1+\varepsilon) \cdot x^y \cdot (2^{1024})^{2\varepsilon+\varepsilon^2} \\
&< (1+\varepsilon) \cdot x^y,
\end{aligned}$$

with $\varepsilon = 2^{-41}$. Similarly, when $\log x$ is within the scope of positive denormalized numbers, it is obtained $\widehat{z} \le (1+\varepsilon) \cdot x^y \cdot (x^y)^\varepsilon \cdot \exp(y \cdot \delta \cdot (1+\varepsilon)) < (1+\varepsilon) \cdot x^y \cdot (2^{1024})^\varepsilon \cdot \exp(2^{1024} \cdot \delta \cdot (1+\varepsilon)) < (1+\varepsilon) \cdot x^y$. If $\log x$ lies between normalized numbers again, but $y \cdot \triangle \log x$ is a positive denormalized number then $\widehat{z} \le (1+\varepsilon) \cdot x^y \cdot (x^y)^\varepsilon \cdot \exp \delta < (1+\varepsilon) \cdot x^y \cdot (2^{1024})^\varepsilon \cdot \exp \delta < (1+\varepsilon) \cdot x^y$. Now, let both $\log x$ and $y \cdot \triangle \log x$ be within the scope of denormalized numbers. Then, $\widehat{z} \le (1+\varepsilon) \cdot x^y \cdot \exp((y+1) \cdot \delta) < (1+\varepsilon) \cdot x^y \cdot \exp(2^{1024} \cdot \delta) < (1+\varepsilon) \cdot x^y$.

The cases where $y$ is negative and/or $\log x$ is negative and/or downward rounding is regarded can be handled analogically, if $\widehat{z}$ is a normalized number. Hence, $\varepsilon$ is the worst-case relative rounding error here. □

---

The inverse images of pow1 on real intervals $\boldsymbol{z}$ can be divided into eight groups $A$–$H$ of relevant states that the relation $\boldsymbol{z} \circledcirc [0,1]$ takes. Further discrimination is necessary in some cases where $\boldsymbol{z}$ may be either unbound or a point interval which comprises a single real number. All cases are listed in Figure 3.2 graphically.

Theorem 3.7. *Tables B.1–B.2 present relevant results of the binary interval reverse power operations for* pow1*.*

Proof. Results can easily be obtained, as mentioned above, by looking at the graphs of inverse images of pow1 in Figure 3.2. Thus, representative of all the many different cases only two of them are formally proven here.

1. Case $\text{pow1}_1^-(\boldsymbol{y},\boldsymbol{z})$ with $\boldsymbol{y}$ afterP $[0,0]$ and $\boldsymbol{z}$ finishes $[0,1]$. It is $0 < \underline{y} \le \overline{y}$ and $0 < \underline{z} \le \overline{z} = 1$. Let $x$ be a positive base with $x > 1$. Then, for all positive $y$ it follows with Equation 2.19 $x^y > x^0 = 1$. Hence, particularly for any $y \in \boldsymbol{y} \subset \mathbb{R}^+$ it is $x^y \notin \boldsymbol{z} = [\underline{z}, 1]$. Now, let $x$ be positive and smaller than $\underline{z}^{1/\underline{y}}$. It holds, using Equation 2.18, $\underline{z}^{1/\underline{y}} \le \underline{z}^0 = 1$, and for all $y \in \boldsymbol{y}$, with Equation 2.21, $x^y \le x^{\underline{y}} < (\underline{z}^{1/\underline{y}})^{\underline{y}} = \underline{z}$. Hence, $x^y \notin \boldsymbol{z} = [\underline{z}, 1]$. Lastly, assume $x \in [\underline{z}^{1/\underline{y}}, 1]$. Then, because of the monotony of the power function it is $\underline{y} \in \boldsymbol{y}$ and $x^{\underline{y}} \in [(\underline{z}^{1/\underline{y}})^{\underline{y}}, 1^{\underline{y}}] = [\underline{z}, 1] = \boldsymbol{z}$.
2. Case $\text{pow1}_2^-(\boldsymbol{x},\boldsymbol{z})$ with $\boldsymbol{x}$ starts $[0,1]$ and $\boldsymbol{z}$ overlappedBy $[0,1]$, $\boldsymbol{z}$ being bounded. Without loss of generality it can be assumed $0 < \overline{x}$ (otherwise the result was empty, because $\text{pow1}(0, y)$ is undefined for any exponent $y$). It is $0 = \underline{x} < \overline{x} < 1$ and $0 < \underline{z} < 1 < \overline{z}$. Let $y$ be an exponent with $y > \log_{\overline{x}} \underline{z}$. Because of $\underline{z} < 1$ the exponent $y$ is positive. Hence, for any $x \in \boldsymbol{x}$ it is $x^y \le \overline{x}^y < \overline{x}^{\log_{\overline{x}} \underline{z}} = \exp(\log_{\overline{x}} \underline{z} \cdot \log \overline{x}) = \exp \log \underline{z} = \underline{z}$, so $x^y \notin \boldsymbol{z}$. Now, let $y < \log_{\overline{x}} \overline{z}$. Analogically, for any base $x \in \boldsymbol{x}$ it follows $x^y \notin \boldsymbol{z}$. Lastly, assume $y \in [\log_{\overline{x}} \overline{z}, \log_{\overline{x}} \underline{z}]$. Then, because of the monotony of the power function, it is $\overline{x} \in \boldsymbol{x}$ and $\overline{x}^y \in [\overline{x}^{\log_{\overline{x}} \underline{z}}, \overline{x}^{\log_{\overline{x}} \overline{z}}] = [\underline{z}, \overline{z}] = \boldsymbol{z}$. □

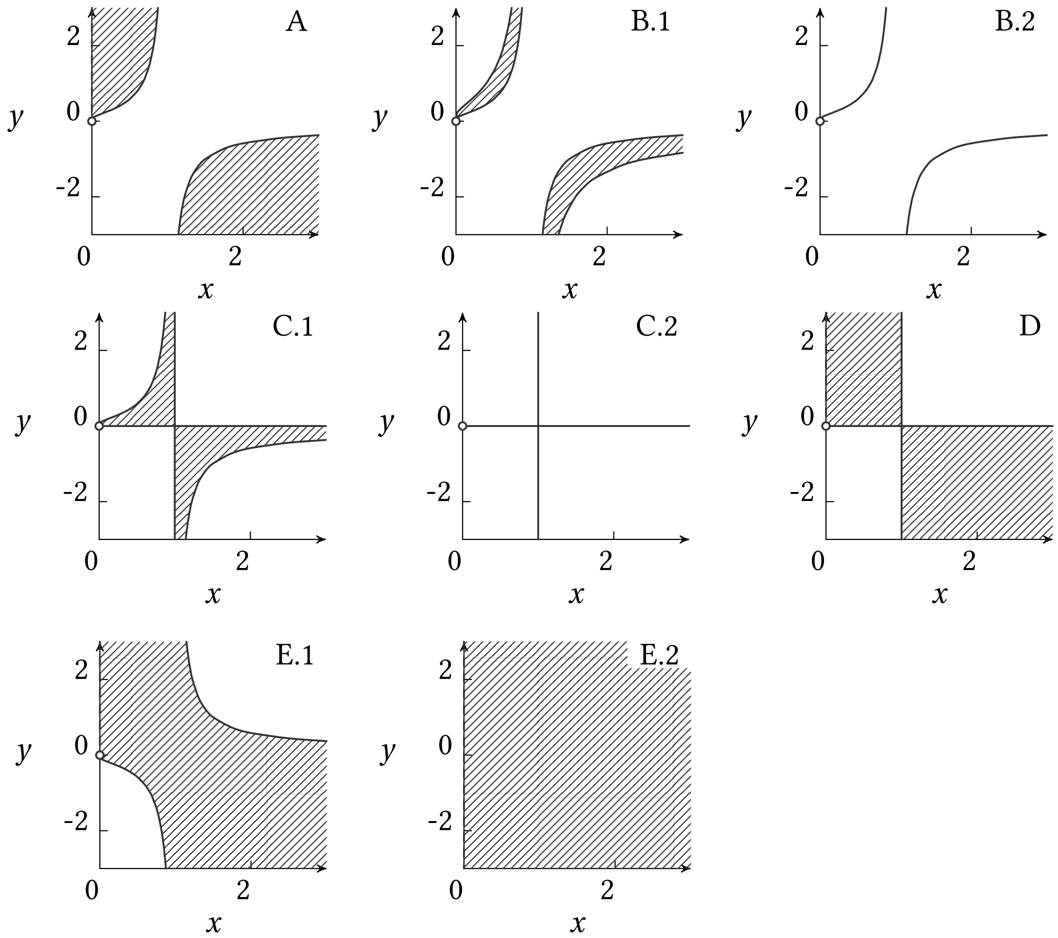


Figure 3.2. Sketches of (two-dimensional) inverse images of intervals of pow1 : $\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto x^y$. $\boldsymbol{z} = [\underline{z}, \overline{z}]$ is a non-empty interval in the codomain with $0 < \overline{z}$, its inverse image is hatched and outlined. Cases are grouped by different states $\boldsymbol{z} ⚭ [0, 1]$. *A*, overlaps/starts. *B.1*, containedByP with $\underline{z} < \overline{z}$. *B.2*, containedByP with $\underline{z} = \overline{z}$. *C.1*, finishes with $\underline{z} < 1$. *C.2*, finishes with $\underline{z} = 1$. *D*, equalP/finishedBy. *E.1*, containsP/startedBy with $\overline{z} < +\infty$. *E.2*, containsP/startedBy with $\overline{z} = +\infty$.

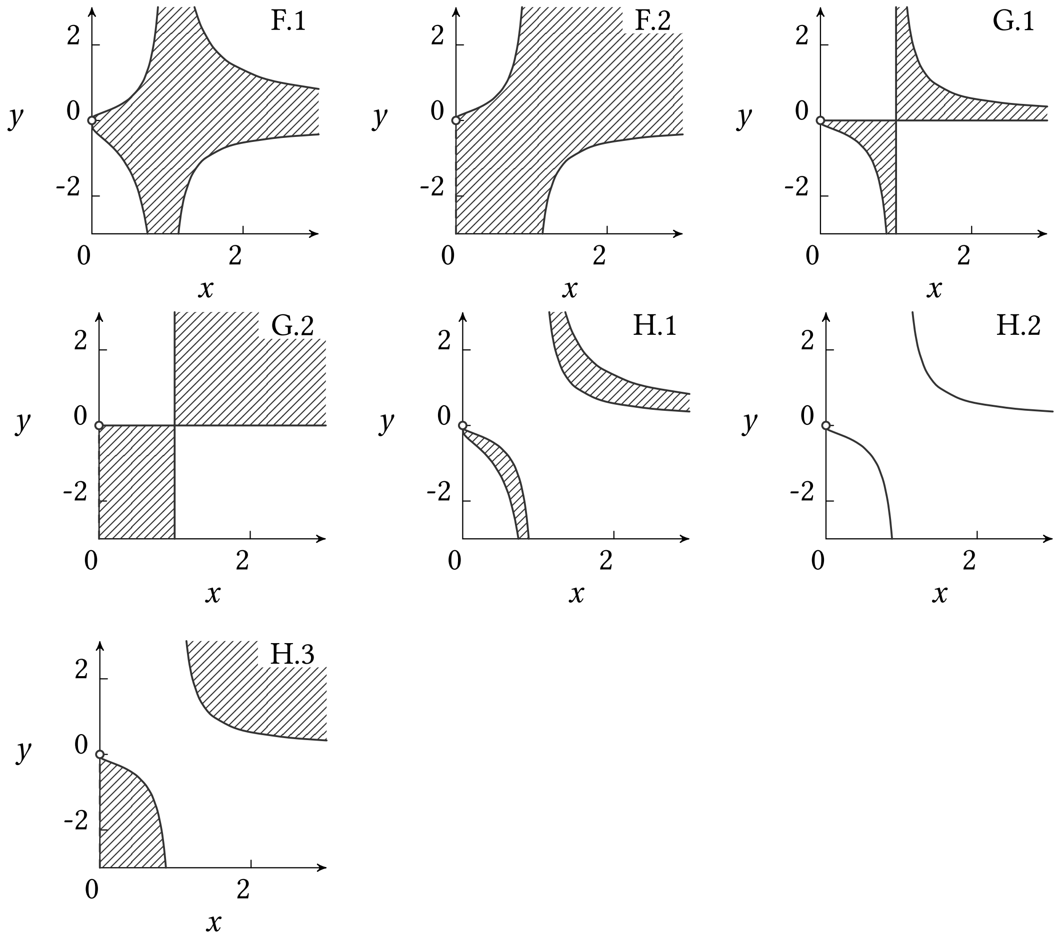


Figure 3.2 *(continued)*. *F.1*, overlappedBy with $\overline{z} < +\infty$. *F.2*, overlappedBy with $\overline{z} = +\infty$. *G.1*, metBy with $\overline{z} < +\infty$. *G.2*, metBy with $\overline{z} = +\infty$. *H.1*, afterP with $\underline{z} < \overline{z} < +\infty$. *H.2*, afterP with $\underline{z} = \overline{z} < +\infty$. *H.3*, afterP with $\overline{z} = +\infty$.

### 3.2.2 Limited Version

Next, I propose the interval extension of the general real power function developed in Chapter 2

$$\begin{aligned} \mathrm{pow2} :& (\mathbb{R}^+ \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}^+) \cup (\mathbb{R}^- \times \mathbb{Z}) \to \mathbb{R}, \\ & (x, y) \mapsto \begin{cases} \exp(y \cdot \log x) & \text{if } x \text{ positive,} \\ 0 & \text{if } x \text{ zero,} \\ \exp(y \cdot \log |x|) & \text{if } x \text{ negative and } y \text{ even,} \\ \text{-}\exp(y \cdot \log |x|) & \text{if } x \text{ negative and } y \text{ odd.} \end{cases} \end{aligned}$$

This version, which is called “Limited Version” in Table 2.2 and which basically is a restriction of the “Complex Version” to the range of real numbers, defines powers for all pairs of base and exponent where sensible and where the result is commonly approved. Positive powers of zero are defined and equal zero, which is valid because of continuity. pow2 equals pow1 for the subset of its domain with positive bases. In short, pow2 is more or less a union of pow1, monomials and reciprocal monomials $x \mapsto x^n$ $(n \in \mathbb{Z})$, if cases with base zero are ignored.

In the following, it suffices to show differences between both power functions (or their interval extensions) in cases where the base may be non-positive. Algorithms A.3–A.5 illustrate how merging with pow1 can be done in practice and how the interval mappings pow2, $\mathrm{pow2}_1^-$ and $\mathrm{pow2}_2^-$ can be evaluated.

Inverse images of pow2, which are shown in Figure 3.3, are more complicated than those of pow1 and require distinction between a lot more cases with the ꝏ relation defined in Subsection 3.1.2. However, in most cases the reverse interval operations $\mathrm{pow2}_1^-$, $\mathrm{pow2}_2^-$ produce results which can simply be computed as the hull of one or two intervals which are possibly to be intersected with the subset of even or odd integral numbers, see Table B.3 and Table B.4. But, for $\mathrm{pow2}_1^-$ there are some cases where the result is a union of infinitely many and possibly disjoint intervals, e.g.,

$$\begin{aligned} \mathrm{pow2}_1^-([1, +\infty], [2, 3]) &= \mathrm{pow1}_1^-([1, +\infty], [2, 3]) \cup \bigcup_{\substack{n \in [1, +\infty] \\ n \text{ even}}} [\text{-}3^{1/n}, \text{-}2^{1/n}] \\ &\subseteq \mathrm{pow1}_1^-([1, +\infty], [2, 3]) \cup \\ &\quad \underbrace{[\text{-}1.74, \text{-}1.41] \cup [\text{-}1.32, \text{-}1.18]}_{\text{disjoint}} \cup \ldots, \end{aligned}$$

cf. Table B.3. In these cases it can be difficult to determine a sharp enclosure of $\mathrm{pow2}_1^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \mathrm{hull}(\boldsymbol{x} \cap \mathrm{pow2}_1^-(\boldsymbol{y}, \boldsymbol{z}))$. In the above example this can be seen for $\boldsymbol{x} = [\text{-}1.38, \text{-}1.2]$, where $\boldsymbol{x}$ needed to be intersected with the interval $[\text{-}1.32, \text{-}1.18]$ only. As it is impossible to compute the intersection of $\boldsymbol{x}$ with the union of possibly

TABLE 3.4. The value of pow2($[\underline{x}, \overline{x}], [\underline{y}, \overline{y}]$) with $\overline{x} < 0$ and $[\underline{y}, \overline{y}]$ comprises at least two integral numbers.

| | $\overline{x} \leq -1$ | $\underline{x} < -1 < \overline{x}$ | $-1 \leq \underline{x}$ |
|---|---|---|---|
| $0 \leq \underline{y}$ | $[\underline{x}^{goe}, \underline{x}^{gee}]$ | $[\underline{x}^{goe}, \underline{x}^{gee}]$ | $[\underline{x}^{loe}, \underline{x}^{lee}]$ |
| $\underline{y} \leq -1$ and $1 \leq \overline{y}$ | $[\underline{x}^{goe}, \underline{x}^{gee}]$ | $\text{hull}([\underline{x}^{goe}, \underline{x}^{gee}] \cup [\overline{x}^{loe}, \overline{x}^{lee}])$ | $[\overline{x}^{loe}, \overline{x}^{lee}]$ |
| $\overline{y} \leq 0$ | $[\overline{x}^{goe}, \overline{x}^{gee}]$ | $[\overline{x}^{loe}, \overline{x}^{lee}]$ | $[\overline{x}^{loe}, \overline{x}^{lee}]$ |

*Note:* Within the interval $[\underline{y}, \overline{y}]$ it is denoted *lee*: lowest even exponent, *gee*: greatest even exponent, *loe*: lowest odd exponent, *goe*: greatest odd exponent, all four numbers exist and are well-defined. In most cases only one boundary of the interval $[\underline{x}, \overline{x}]$ takes part. In these cases it suffices to compute a single logarithm. For the cases with $\underline{x} = -\infty$ or $\overline{x} = 0$ one can apply limiting values of above results. Cases, where $[\underline{y}, \overline{y}]$ contains only a single integral number, are dealt with in Table 3.5.

TABLE 3.5. The value of pow2($[\underline{x}, \overline{x}], \{n\}$) with $\overline{x} < 0$ and $n \in \mathbb{Z}$.

| | $n$ even | $n$ odd |
|---|---|---|
| $0 \leq n$ | $[\overline{x}^n, \underline{x}^n]$ | $[\underline{x}^n, \overline{x}^n]$ |
| $n \leq 0$ | $[\underline{x}^n, \overline{x}^n]$ | $[\overline{x}^n, \underline{x}^n]$ |

*Note:* For the cases with $\underline{x} = -\infty$ or $\overline{x} = 0$ one can apply limiting values of above results.

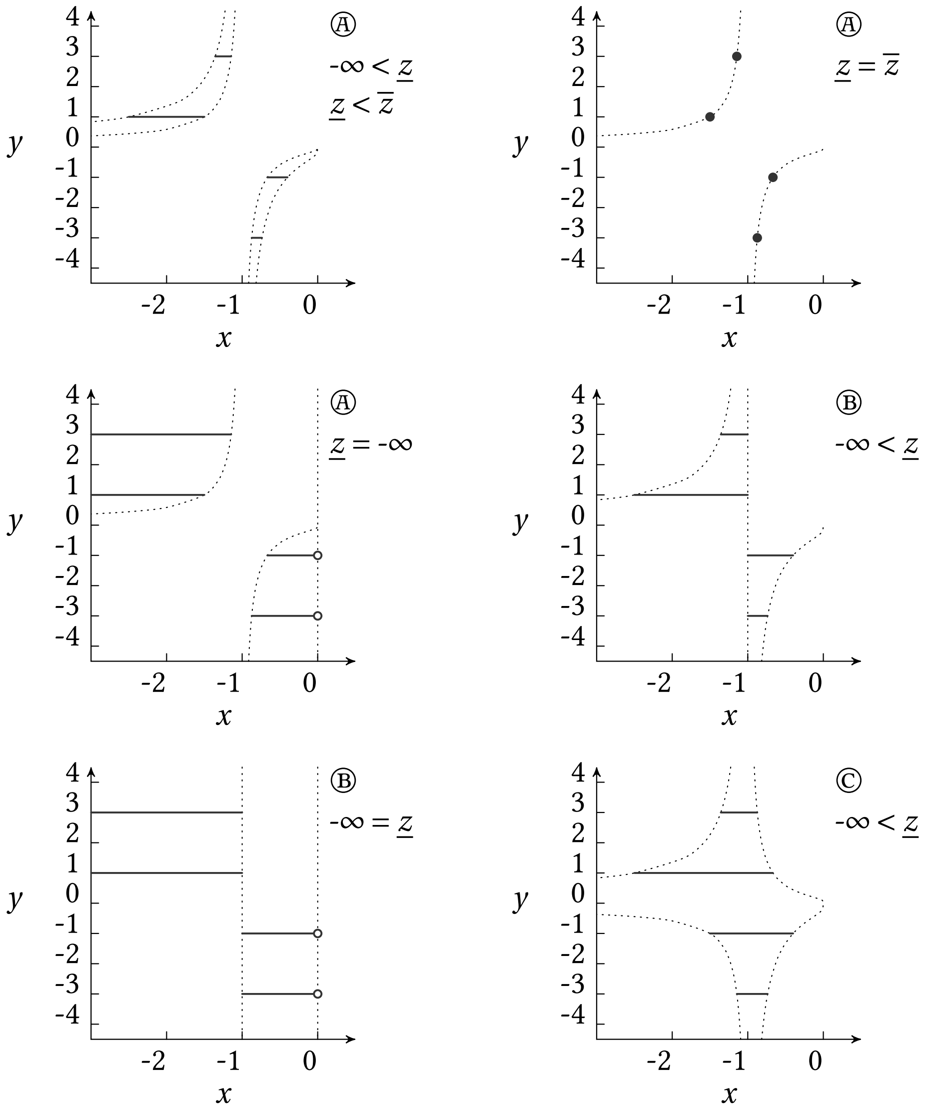


FIGURE 3.3. Sketches of (two-dimensional) inverse images of intervals of pow2 : $(x, y) \mapsto x^y$. $\boldsymbol{z} = [\underline{z}, \overline{z}]$ is a non-empty interval in the codomain, its reverse image is drawn. Cases are grouped by different states of $\boldsymbol{z}$∞∞$[-1, 1]$, see detachable card as an appendix for examples.

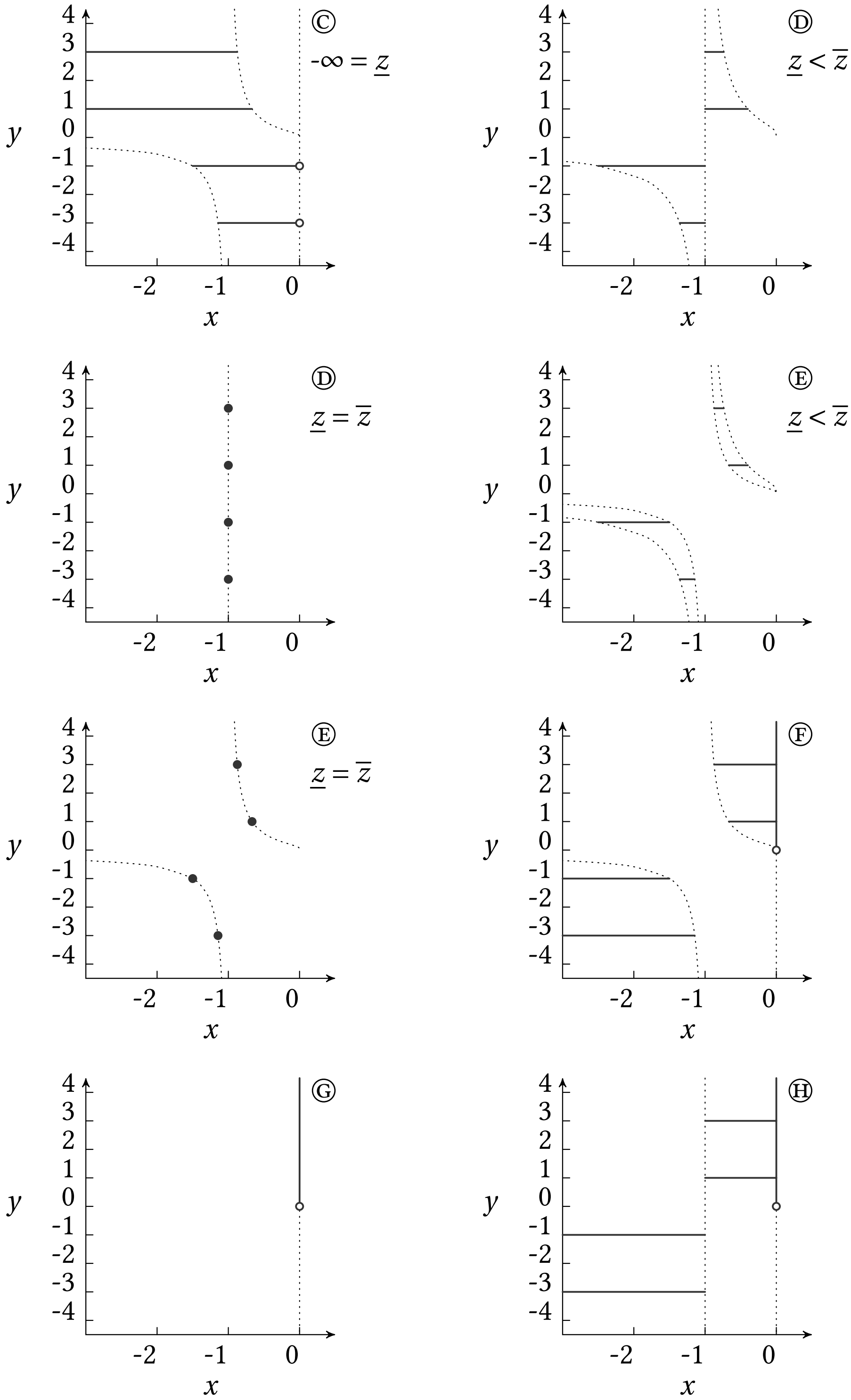


FIGURE 3.3 *(CONTINUED)*

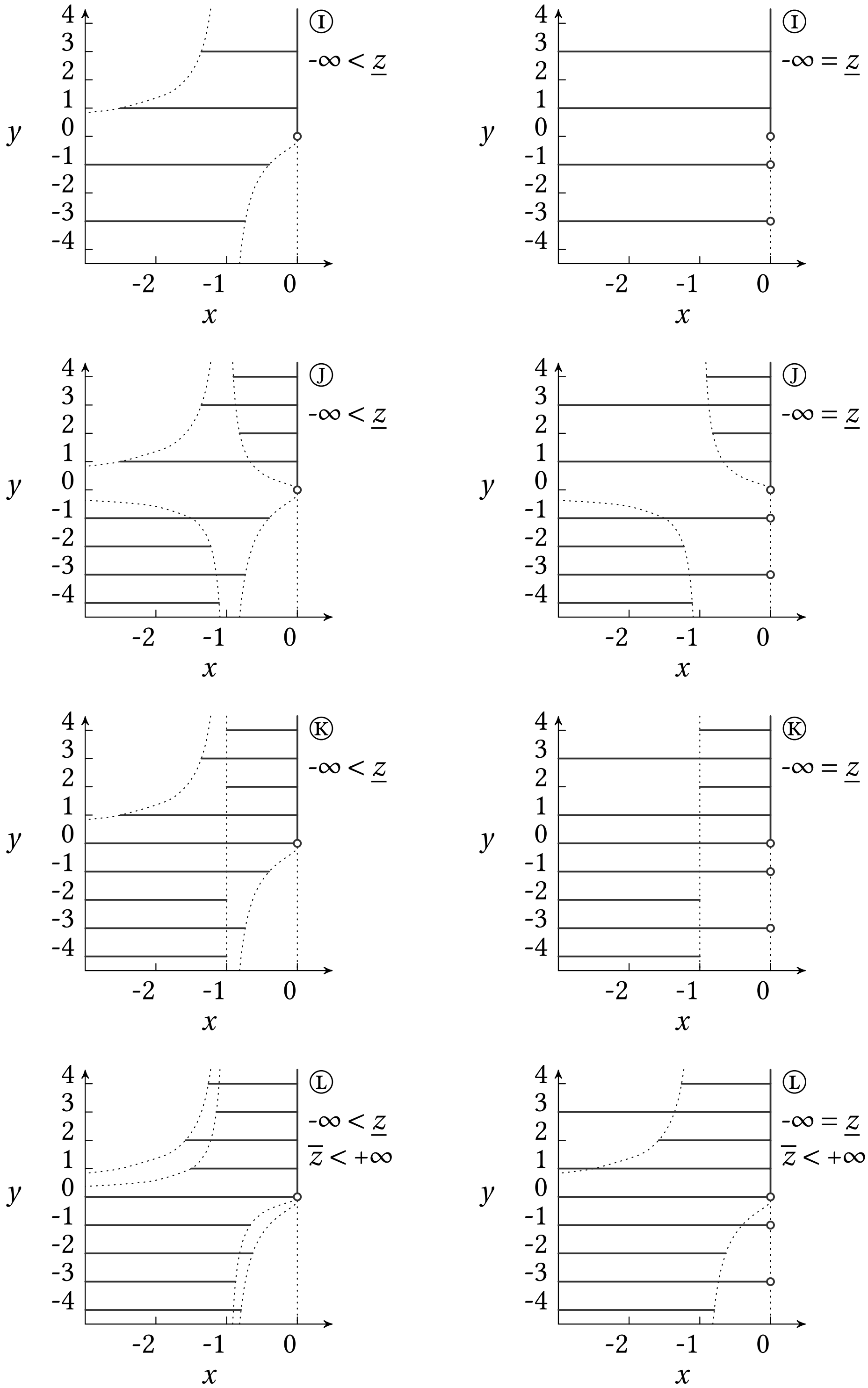


FIGURE 3.3 *(CONTINUED)*

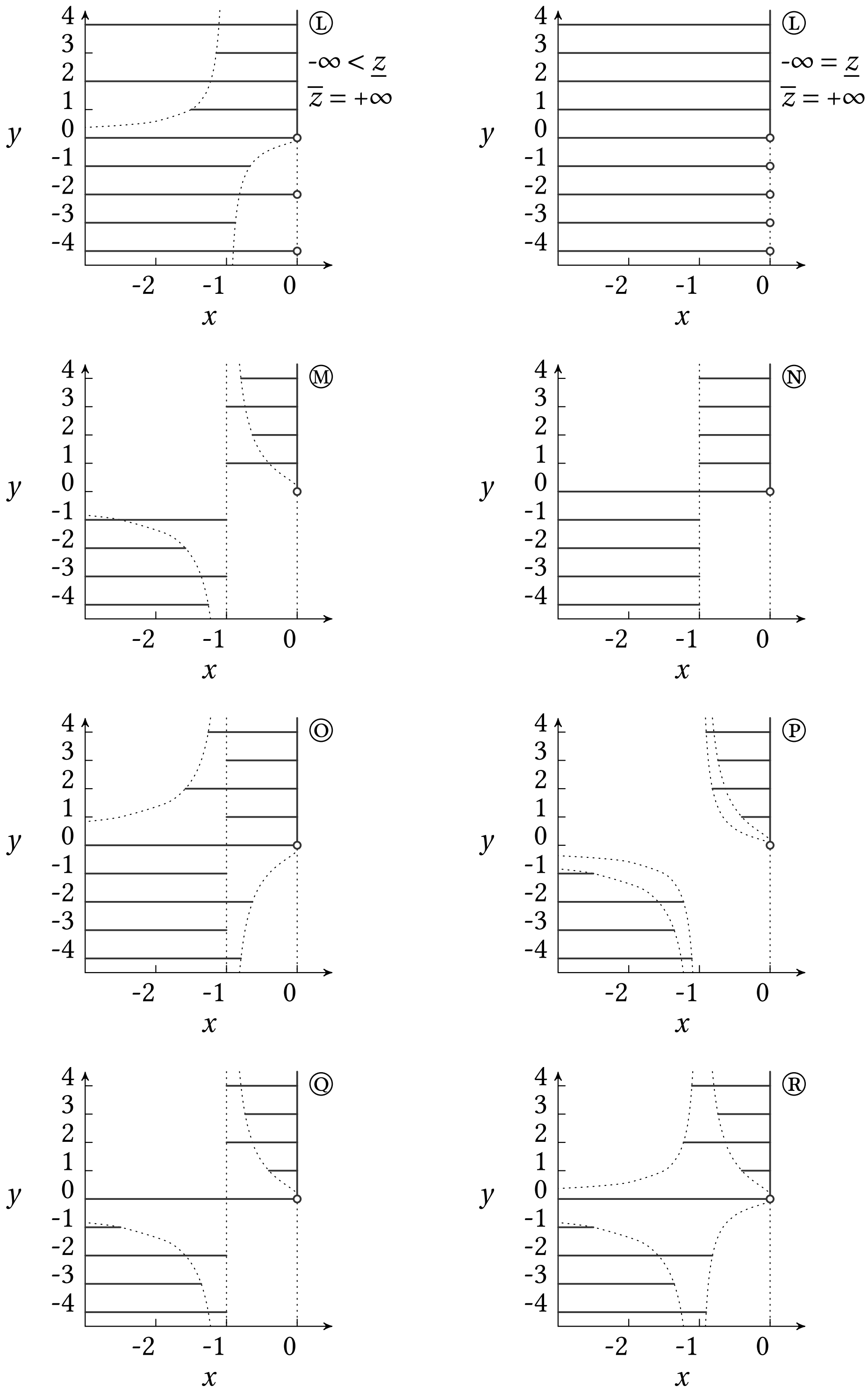


FIGURE 3.3 *(CONTINUED)*

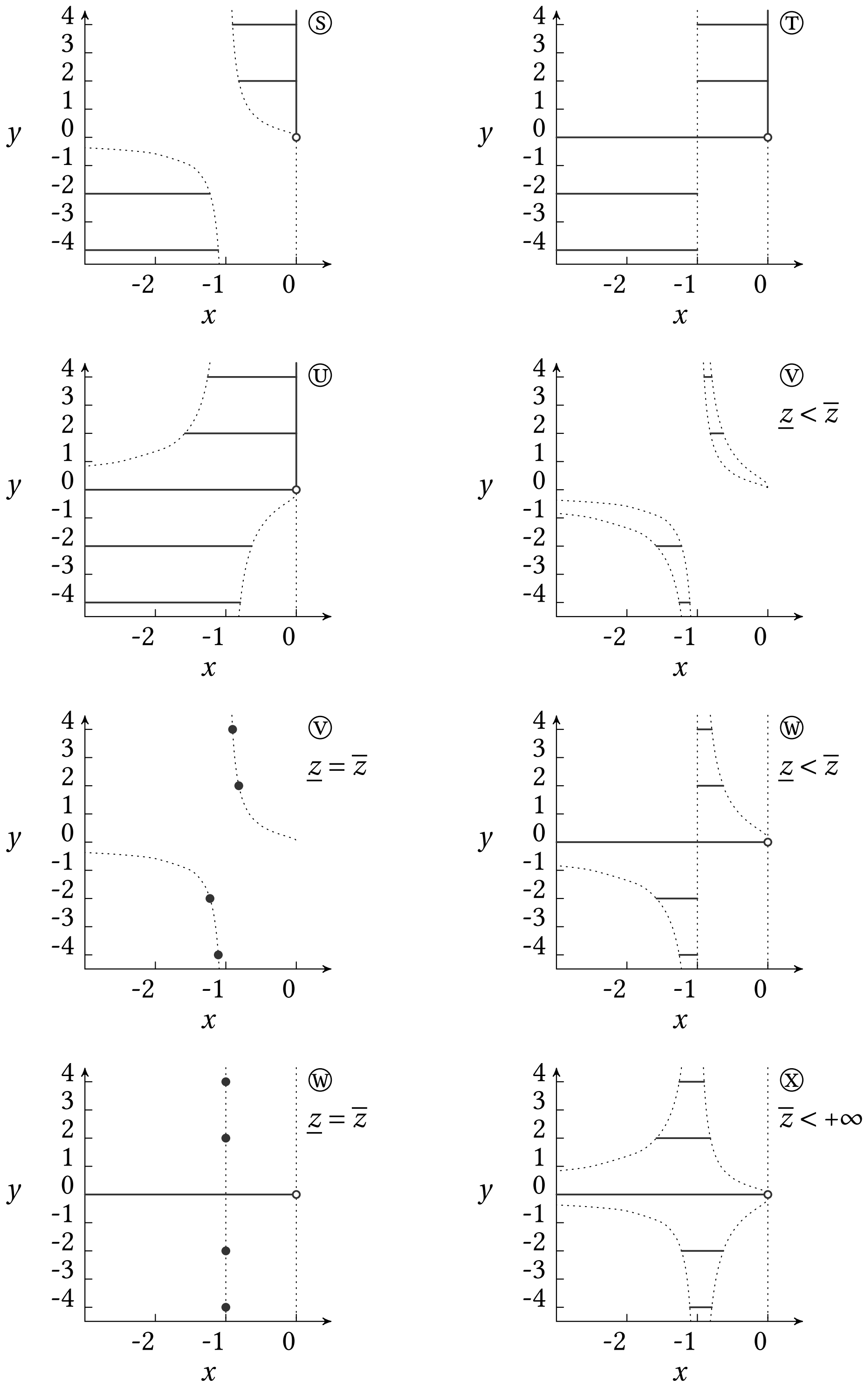


FIGURE 3.3 *(CONTINUED)*

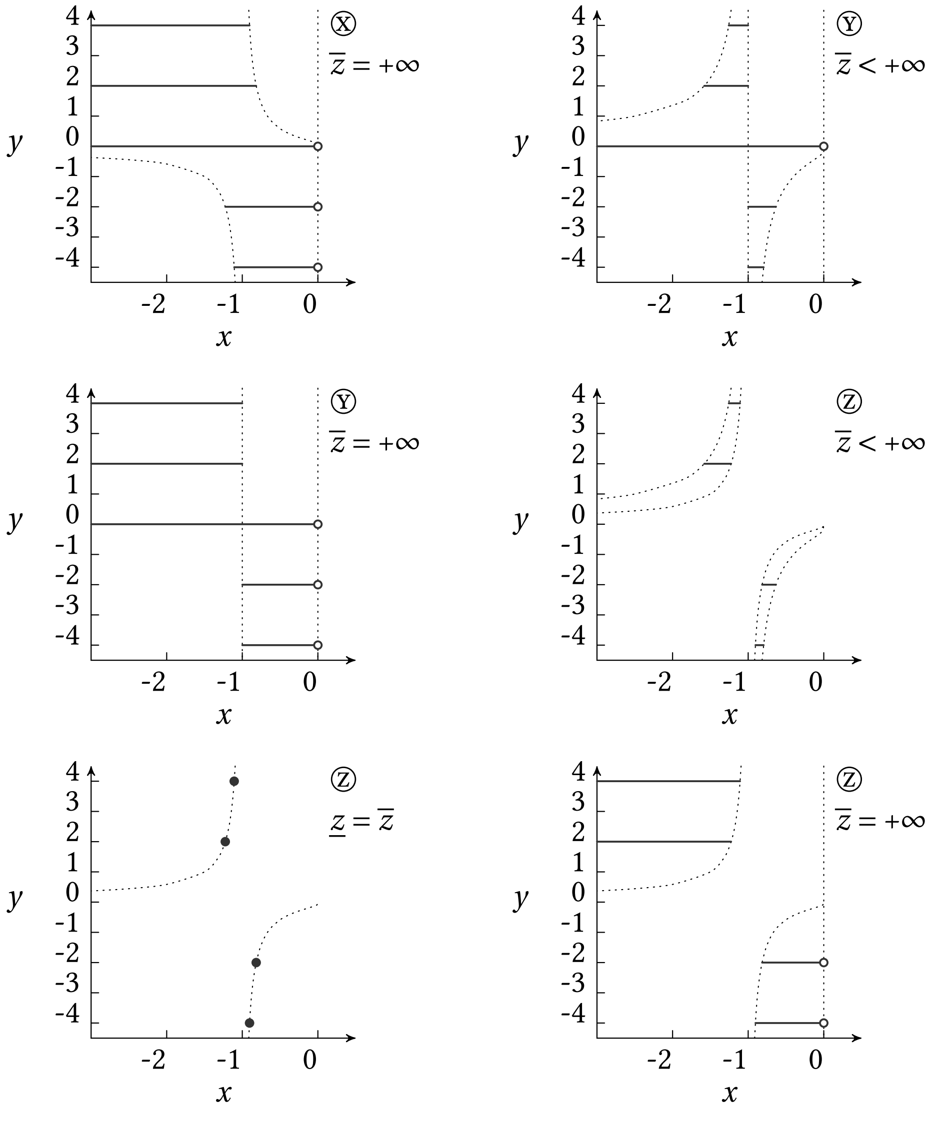


Figure 3.3 *(continued)*

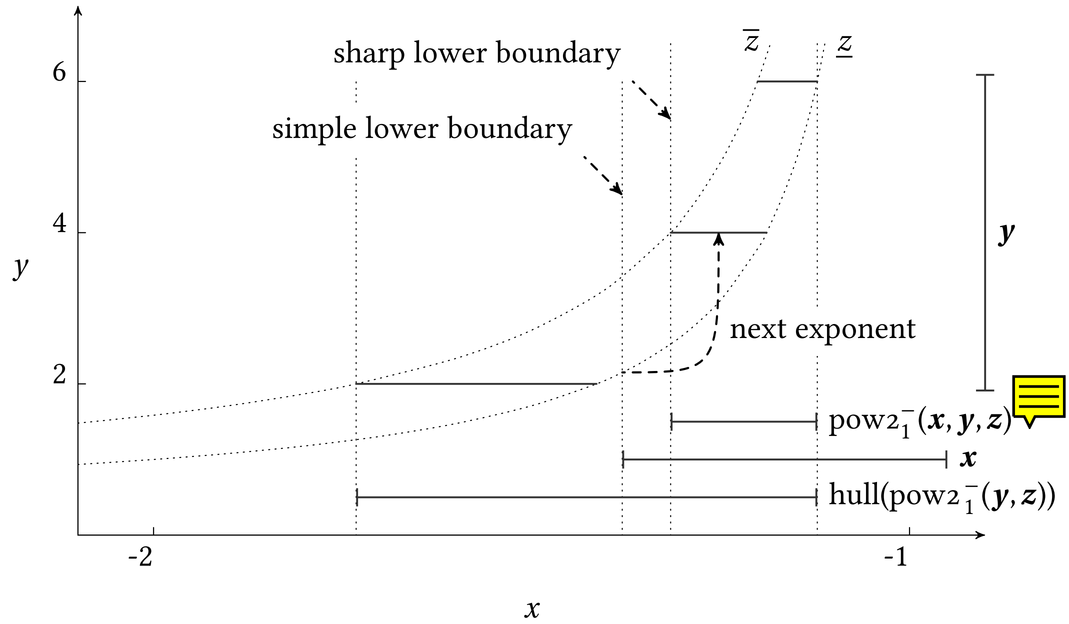


FIGURE 3.4. Computation of $\mathrm{pow2}_1^-$ in cases where $\overline{x} \leq 0$, $\underline{y} > 0$ and $\underline{z} > 1$. The result cannot be computed directly, but together with $\mathrm{hull}(\mathrm{pow2}_1^-(\boldsymbol{y},\boldsymbol{z}))$ one can obtain a simple containment. Finding the next exponent, which is—together with a member from $\boldsymbol{x}$—part of pow2's inverse image, subsequently leads to the sharp boundaries, see Algorithm A.6. In the shown example it is $\boldsymbol{z} = [2, 3]$.

an infinite number of intervals one by one, I propose Algorithm A.6, which—with minor changes—can be applied to all difficult cases, i.e., where $\boldsymbol{z}$ ∞⦶ $[-1, 1]$ equals Ⓐ, Ⓔ, Ⓥ or Ⓩ .

Algorithm A.6 works as follows: At first, an enclosure of the union of the many intervals is computed, which, when intersected with $\boldsymbol{x}$, already produces an enclosure of the result. Each boundary of this enclosure is sharp if, and only if, it is part of the union of many intervals $U = \mathrm{pow2}_1^-(\boldsymbol{y},\boldsymbol{z})$. Thus, the result's boundaries only need to be optimized if they are not part of the reverse operation's result $U$. At this point, the algorithm utilizes that the relevant part of the inverse image of $\boldsymbol{z}$ consists of individual lines which are parallel to the $x$ axis, i.e., $\{(x, y) \mid x \leq 0 \text{ and } \mathrm{pow2}(x, y) \in \boldsymbol{z}\} \subset \mathbb{R} \times \mathbb{Z}$, cf. Figure 3.3. For the lower boundary, the algorithm determines the exponent $b$ for the lowest interval from $U$ that takes part in the result, i.e., there exists $x \in \boldsymbol{x}$ with $\mathrm{pow2}(x, b) \in \boldsymbol{z}$. Because $b$ belongs to the lowest relevant interval $I$ in $U$, there are two possibilities: If $I$'s left boundary is lower than $\underline{x}$, $\underline{x}$ is contained in $U$ and the lower boundary of the result is sharp. On the other side, if $I$'s left boundary is greater than $\underline{x}$, $\underline{x}$ is not contained in $U$ and the result is not sharp. The left boundary of $I$ is denoted by $c$ in the algorithm and is a proper boundary for $U \cap \boldsymbol{x}$, which additionally is sharp for $\underline{x} < c$. An analogous optimization can be made with the upper boundary. The idea behind the algorithm is illustrated in Figure 3.4 graphically.

A floating-point version of the algorithm is no more difficult. Correct upward/-downward directed rounding needs to be applied during calculation of the variables $a$, $b$ and $c$ in Algorithm A.6.

### 3.2.3 Extended Version

Admission of an extended version of the power function for the upcoming standard for interval arithmetic is currently under discussion. In the last chapter I do not recommend using this variant, which is called "Alternative Version" in Table 2.2, for what shall be known as the *general power function* due to a questionable mathematical foundation, but, like the work of Averbukh and Günther mentioned in Section 2.4.2, there exist applications where this variant can be of great use.

In the following, the possibilities of

$$\text{pow3} : (\mathbb{R}^+ \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}^+) \cup (\mathbb{R}^- \times \{\tfrac{m}{n} \mid m, n \in \mathbb{Z},\ n \text{ odd}\}) \to \mathbb{R},$$
$$(x, y) \mapsto \begin{cases} \exp(y \cdot \log x) & \text{if } x \text{ positive,} \\ 0 & \text{if } x \text{ zero,} \\ \exp(y \cdot \log|x|) & \text{if } x \text{ negative and } y = \frac{m}{n} \text{ with } m \text{ even} \\ & \qquad \text{and } n \text{ odd,} \\ \text{-}\exp(y \cdot \log|x|) & \text{if } x \text{ negative and } y = \frac{m}{n} \text{ with } m \text{ odd} \\ & \qquad \text{and } n \text{ odd,} \end{cases}$$

for interval arithmetic are developed and discussed. There are several remarkable points about this function, which virtually call for a use of pow3 in interval arithmetic. First of all, as written in Section 2.4.2, the mathematical definition of this function is problematic in a floating-point context. In a floating-point context with base 2, which is the common case, things are worse: All finite numbers can be written as $n/2^e$ with $n, e \in \mathbb{Z}$ and respective bounds for $n$ and $e$, thus, non-integral numbers are rational and can only be written as a fraction with even denominator (the denominator is a power of 2 for reduced fractions). Hence, pow3 can only be computed on the very same domain where pow2 is defined, and there is no difference between them in floating-point. Accessorily, there is no difference between interval extensions of pow2 and pow3 when evaluated for exponents being point-intervals containing only a single floating-point number with base 2.

In practice, pow3 defines powers like for example $(-8)^{1/3} = -2$ or $(-8)^{2/3} = 4$, but they can not be computed directly, due to representational problems with rational exponents. However, with interval arithmetic one can overcome this problem by using an interval enclosure of the exponent, e.g., $(-8)^{\text{hull}(\{1/3\})}$, which will then contain the exact result -2. Yet, the enclosure of the result is of minor benefit only, as it must also contain 2, because of those many fractions with even nominator in hull({1/3}). If such powers shall be computed with the exponent being a single rational number, it is highly recommended to use the different function powr : $(x, m, n) \mapsto x^{m/n}$, which is briefly mentioned at the beginning of this chapter and which resolves the representational problems of radical exponents.

---

In the past, there have been different suggestions regarding the definition of pow3. On the domain of negative bases, the subset of exponents for which pow3 is defined is dense in $\mathbb{R}$ (a proof is given in Lemma 3.9 below), and could therefore serve as a basis for a continuous extension of the function. However, $\pm\exp(y \cdot \log|x|)$ are both accumulation points of pow3 at $(x, y)$ with $x < 0$, because there are fractions with odd denominators and even as well as odd nominators in every neighborhood of $y$. Alternatively, one could evaluate $\exp(y \cdot \log x)$ with complex versions of exp and log with infinite branches of the complex logarithm. For $x < 0$, the complex results are becoming arbitrary close to the reals at both values $\pm\exp(y \cdot \log|x|)$ (Dan Zuras, pers. comm.). Either way, looking at real accumulation points of pow3, results arise together with both arithmetic signs at each point, which is an important property and something that would be of major importance in containment set theory (Pryce and Corliss 2006).

But, *does it justify the definition of a (two-valued) function* $\widetilde{\text{pow3}}$ *to be used in interval arithmetic instead of* pow3*?* It can be defined, and it has been proposed by Dan Zuras to the interval standard working group,

$$\widetilde{\text{pow3}} : (\mathbb{R}^+ \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}^+) \cup (\mathbb{R}^- \times \mathbb{R}) \to \wp(\mathbb{R})$$
$$(x, y) \mapsto \begin{cases} \{\exp(y \cdot \log x)\} & \text{if } x \text{ positive,} \\ \{0\} & \text{if } x \text{ zero,} \\ \{\pm\exp(y \cdot \log|x|)\} & \text{if } x \text{ negative,} \end{cases}$$

an interval version of which would be simple to compute (ibid.). Although it is very clear how Definition 3.2 could be applied to $\widetilde{\text{pow3}}$ in a reasonable way to make an interval function $\widetilde{\text{pow3}}$—even reverse operations could be defined with minor adjustments—, it is a big issue that this function is a no point-function. Especially during the evaluation of $\widetilde{\text{pow3}}$ on point intervals for the exponent, the result is too extensive, if one is not interested in the limit values that are introduced by the two-valued function.

However, the function $\widetilde{\text{pow3}}$ has an advantage over pow3: Its "interval extension" can be computed more easily than pow3 directly, with fewer cases to consider and with the help of the positive version pow1.

Lemma 3.8. *For negative base intervals $\boldsymbol{x}$ and exponent intervals $\boldsymbol{y}$ it holds*

$$\widetilde{\mathit{pow3}}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{hull}\{\pm \mathit{pow1}(\text{-}\boldsymbol{x},\boldsymbol{y})\} = \mathrm{hull}(\text{-}\mathit{pow1}(\text{-}\boldsymbol{x},\boldsymbol{y}) \cup \mathit{pow1}(\text{-}\boldsymbol{x},\boldsymbol{y})). \tag{3.3}$$

Proof. It is assumed that the value of $\widetilde{\mathrm{pow3}}$ is defined as the hull of the union of all results $\widetilde{\mathrm{pow3}}(x,y)$ with $x \in \boldsymbol{x}$ and $y \in \boldsymbol{y}$ being parameters in the function's domain, cf. Definition 3.2. The lemma follows directly from the identity $\widetilde{\mathrm{pow3}}(x,y) = \{\pm \exp(y \cdot \log \text{-}x)\} = \{\text{-}\mathrm{pow1}(\text{-}x,y)\} \cup \{\mathrm{pow1}(\text{-}x,y)\}$ and the closure of the hull operator. □

Results for $\widetilde{\mathrm{pow3}}$ can therefore be given easily. From the definition of $\widetilde{\mathrm{pow3}}$ and the fact, that pow1 ignores negative bases, an easy equation for computation of the function is obtained,

$$\widetilde{\mathrm{pow3}}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{hull}\{\pm \mathrm{pow1}(\text{-}\boldsymbol{x},\boldsymbol{y}), \mathrm{pow1}(\boldsymbol{x},\boldsymbol{y})\} \tag{3.4}$$

holds for arbitrary intervals $\boldsymbol{x}$ and $\boldsymbol{y}$, if $\boldsymbol{x} \neq \{0\}$. Furthermore, $\widetilde{\mathrm{pow3}}$ can actually be used instead of pow3 in many cases.

Lemma 3.9. *For base intervals $\boldsymbol{x}$ and exponent intervals $\boldsymbol{y} = [\underline{y}, \overline{y}]$ with $\underline{y} < \overline{y}$ it holds* $\mathit{pow3}(\boldsymbol{x},\boldsymbol{y}) = \widetilde{\mathit{pow3}}(\boldsymbol{x},\boldsymbol{y})$.

Proof. Let $[\underline{z}_0, \overline{z}_0] = \mathrm{pow3}(\boldsymbol{x},\boldsymbol{y})$ and $[\underline{z}_1, \overline{z}_1] = \widetilde{\mathrm{pow3}}(\boldsymbol{x},\boldsymbol{y})$ both being non-empty. Without loss of generality it can be assumed that $\underline{z}_0$ and $\underline{z}_1$ are both negative and $\overline{z}_0$ and $\overline{z}_1$ are both positive (otherwise pow3 and $\widetilde{\mathrm{pow3}}$ were evaluated on the domain of non-negative bases only, on which they agree).

From the definitions of pow3 and $\widetilde{\mathrm{pow3}}$ it follows

$$\begin{aligned}
\underline{z}_0 &= \inf\{\text{-}\exp(\tfrac{m}{n} \cdot \log|x|) \mid x \in \boldsymbol{x} \text{ negative and } \tfrac{m}{n} \in \boldsymbol{y} \text{ with } m \text{ odd and } n \text{ odd}\},\\
\underline{z}_1 &= \inf\{\text{-}\exp(y \cdot \log|x|) \mid x \in \boldsymbol{x} \text{ negative and } y \in \boldsymbol{y}\},\\
\overline{z}_0 &= \max(\\
&\qquad \sup\{\exp(\tfrac{m}{n} \cdot \log|x|) \mid x \in \boldsymbol{x} \text{ negative and } \tfrac{m}{n} \in \boldsymbol{y} \text{ with } m \text{ even and } n \text{ odd}\}\\
&\qquad \sup\{\exp(y \cdot \log x) \mid x \in \boldsymbol{x} \text{ positive and } y \in \boldsymbol{y}\}),\\
\overline{z}_1 &= \max(\\
&\qquad \sup\{\exp(y \cdot \log|x|) \mid x \in \boldsymbol{x} \text{ negative and } y \in \boldsymbol{y}\}\\
&\qquad \sup\{\exp(y \cdot \log x) \mid x \in \boldsymbol{x} \text{ positive and } y \in \boldsymbol{y}\}).
\end{aligned}$$

The set of rational numbers is dense in $\mathbb{R}$, as well as the set of rational numbers that can be written as a fraction with an odd denominator: For each two rational numbers $r$ and $s$ with $r = \frac{m}{n} < \frac{k}{n} = s$, where $m$ and $k$ are integral nominators and $n$ is the common positive integral denominator, it can be defined $t = \frac{a}{b}$ with

$a = \lfloor (m+k)\cdot(1+1/2n)\rfloor$ and $b = 2n+1$. Obviously, $t$ is rational and can be written as a fraction with an odd denominator. Additionally, it holds $t \in ]r,s[$, because

$$\begin{aligned} t &= \lfloor (m+k)\cdot(1+1/2n)\rfloor/(2n+1) \\ &\le (m+k)\cdot(1+1/2n)/(2n+1) \\ &= (m+k)/2n \\ &= (r+s)/2 \\ &< s \end{aligned}$$

and

$$\begin{aligned} t &= \lfloor (m+k)\cdot(1+1/2n)\rfloor/(2n+1) \\ &> ((m+k)\cdot(1+1/2n)-1)/(2n+1) \\ &= (m+k)\cdot(1+1/2n)/(2n+1) - 1/(2n+1) \\ &= (m+k)/2n - 1/(2n+1) \\ &= m/n - 2m/2n + (m+k)/2n - 1/(2n+1) \\ &= r + \underbrace{(k-m)}_{\ge 1}/2n - 1/(2n+1) \\ &\ge r + \underbrace{1/2n - 1/(2n+1)}_{>0} \\ &> r. \end{aligned}$$

Then, it can be defined $t = (a+1)/b$, for which $t \in ]r,s[$ holds as well (can be proven analogically). Exactly one of the nominators $a$ and $a+1$ is odd and the other one is even. Hence, the rational numbers $t$ and $t$ show that between each two rational numbers there are rational numbers that can be written with odd denominator and odd or even nominator respectively.

The identities $\underline{z}_0 = \underline{z}_1$ and $\overline{z}_0 = \overline{z}_1$ finally follow from $\underline{y} < \overline{y}$, i.e., the dense subsets of the exponent's range are not empty, together with the continuity of pow1. □

COROLLARY. *Summing up Equation 3.4 and previous lemmas, an interval extension of* pow*3* *can be computed in a floating-point context* $\mathbb{F}$ *with base* 2 *as follows*

$$\mathrm{pow}3_{\mathbb{F}} : \overline{\mathbb{IF}} \times \overline{\mathbb{IF}} \to \overline{\mathbb{IF}}$$

$$(\boldsymbol{x},\boldsymbol{y}) \mapsto \begin{cases} \emptyset & \textit{if } \boldsymbol{y} \subseteq ]\text{-}\infty, 0] \textit{ and } \boldsymbol{x} \subseteq \{0\}, \\ \{0\} & \textit{if } \boldsymbol{y} \cap \mathbb{R}^+ \neq \emptyset \textit{ and } \boldsymbol{x} = \{0\}, \\ \mathrm{pow}2_{\mathbb{F}}(\boldsymbol{x},\boldsymbol{y}) & \textit{if } \underline{y} = \overline{y}, \\ \mathrm{hull}\{\pm \mathrm{pow}1_{\mathbb{F}}(\text{-}\boldsymbol{x},\boldsymbol{y}), \mathrm{pow}1_{\mathbb{F}}(\boldsymbol{x},\boldsymbol{y})\} & \textit{otherwise.} \end{cases}$$

Due to the definition of pow3, its reverse operations can be put together using reverse operations of pow1. Again, it shall be noted that pow1 as well as its reverse operations ignore non-positive values for base $\boldsymbol{x}$ and result $\boldsymbol{z}$.

THEOREM 3.10. *For arbitrary intervals $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ it holds*

$$\begin{aligned}
\mathrm{pow3}_1^-(\boldsymbol{y},\boldsymbol{z}) = \mathrm{pow1}_1^-(\boldsymbol{y},\boldsymbol{z}) & \\
&\cup \begin{cases} \{0\} & \textit{if } \boldsymbol{y} \cap \mathbb{R}^+ \neq \emptyset \textit{ and } 0 \in \boldsymbol{z} \\ \emptyset & \textit{otherwise} \end{cases} \qquad (3.5) \\
&\cup \text{-}\mathrm{pow1}_1^-(\boldsymbol{y}_{even}, \boldsymbol{z}) \\
&\cup \text{-}\mathrm{pow1}_1^-(\boldsymbol{y}_{odd}, \text{-}\boldsymbol{z}), \\
\mathrm{pow3}_2^-(\boldsymbol{x},\boldsymbol{z}) = \mathrm{pow1}_2^-(\boldsymbol{x},\boldsymbol{z}) & \\
&\cup \begin{cases} \mathbb{R}^+ & \textit{if } 0 \in \boldsymbol{x} \textit{ and } 0 \in \boldsymbol{z} \\ \emptyset & \textit{otherwise} \end{cases} \qquad (3.6) \\
&\cup (\mathrm{pow1}_2^-(\text{-}\boldsymbol{x}, \boldsymbol{z}) \cap \mathbb{Q}_{even}) \\
&\cup (\mathrm{pow1}_2^-(\text{-}\boldsymbol{x}, \text{-}\boldsymbol{z}) \cap \mathbb{Q}_{odd}),
\end{aligned}$$

*where*

$$\begin{aligned}
\mathbb{Q}_{even} &= \{r \in \mathbb{Q} \mid r = \tfrac{m}{n} \textit{ with } m \textit{ even and } n \textit{ odd}\}, \\
\mathbb{Q}_{odd} &= \{r \in \mathbb{Q} \mid r = \tfrac{m}{n} \textit{ with } m \textit{ odd and } n \textit{ odd}\}, \\
\boldsymbol{y}_{even} &= \mathrm{hull}(\boldsymbol{y} \cap \mathbb{Q}_{even}) \\
&= \begin{cases} \emptyset & \textit{if } \boldsymbol{y} = \{\frac{m}{n}\} \textit{ with } m \textit{ odd and } n \textit{ odd,} \\ \boldsymbol{y} & \textit{otherwise,} \end{cases} \\
\boldsymbol{y}_{odd} &= \mathrm{hull}(\boldsymbol{y} \cap \mathbb{Q}_{odd}) \\
&= \begin{cases} \emptyset & \textit{if } \boldsymbol{y} = \{\frac{m}{n}\} \textit{ with } m \textit{ even and } n \textit{ odd,} \\ \boldsymbol{y} & \textit{otherwise.} \end{cases}
\end{aligned}$$

PROOF. From Definition 3.5 it follows

$$\begin{aligned}
\mathrm{pow3}_1^-(\boldsymbol{y},\boldsymbol{z}) &= \{x \in \mathbb{R} \mid \text{there exists } y \in \boldsymbol{y} \text{ with } \mathrm{pow3}(x,y) \in \boldsymbol{z}\} \\
&= A_1 \cup A_2 \cup A_3 \cup A_4, \\
\mathrm{pow3}_2^-(\boldsymbol{x},\boldsymbol{z}) &= \{y \in \mathbb{R} \mid \text{there exists } x \in \boldsymbol{x} \text{ with } \mathrm{pow3}(x,y) \in \boldsymbol{z}\} \\
&= B_1 \cup B_2 \cup B_3 \cup B_4,
\end{aligned}$$

and reverse operations can be given using a union of cases, where the results are partitioned using $A_i, B_i$ as follows.

$$\begin{aligned}
A_1 &= \{x \in \mathbb{R}^+ \mid \text{there exists } y \in \boldsymbol{y} \text{ with pow3}(x,y) \in \boldsymbol{z}\},\\
A_2 &= \{x \in \{0\} \mid \text{there exists } y \in \boldsymbol{y} \text{ with pow3}(x,y) \in \boldsymbol{z}\},\\
A_3 &= \{x \in \mathbb{R}^- \mid \text{there exists } y \in \boldsymbol{y} \text{ with pow3}(x,y) \in \boldsymbol{z} \cap \mathbb{R}^+\},\\
A_4 &= \{x \in \mathbb{R}^- \mid \text{there exists } y \in \boldsymbol{y} \text{ with pow3}(x,y) \in \boldsymbol{z} \cap \mathbb{R}^-\},\\
B_1 &= \{y \in \mathbb{R} \mid \text{there exists } x \in \boldsymbol{x} \cap \mathbb{R}^+ \text{ with pow3}(x,y) \in \boldsymbol{z}\},\\
B_2 &= \{y \in \mathbb{R} \mid \text{there exists } x \in \boldsymbol{x} \cap \{0\} \text{ with pow3}(x,y) \in \boldsymbol{z}\},\\
B_3 &= \{y \in \mathbb{R} \mid \text{there exists } x \in \boldsymbol{x} \cap \mathbb{R}^- \text{ with pow3}(x,y) \in \boldsymbol{z} \cap \mathbb{R}^+\},\\
B_4 &= \{y \in \mathbb{R} \mid \text{there exists } x \in \boldsymbol{x} \cap \mathbb{R}^- \text{ with pow3}(x,y) \in \boldsymbol{z} \cap \mathbb{R}^-\}.
\end{aligned}$$

The sets $A_3$ and $A_4$ as well as $B_1$, $B_2$, $B_3$ and $B_4$ not necessarily are disjoint. However, with above restrictions to the domain of $x$ and $z$ respectively, each of the eight sets can be computed individually with less complexity. It follows

$$\begin{aligned}
A_1 &= \{x \in \mathbb{R}^+ \mid \text{there exists } y \in \boldsymbol{y} \text{ with } \exp(y \cdot \log x) \in \boldsymbol{z}\}\\
&= \text{pow1}_1^-(\boldsymbol{y}, \boldsymbol{z}),\\
A_2 &= \{x \in \{0\} \mid \text{there exists } y \in \boldsymbol{y} \cap \mathbb{R}^+ \text{ with } 0 \in \boldsymbol{z}\}\\
&= \begin{cases} \{0\} & \text{if } \boldsymbol{y} \cap \mathbb{R}^+ \neq \emptyset \text{ and } 0 \in \boldsymbol{z},\\ \emptyset & \text{otherwise,} \end{cases}\\
A_3 &= \{x \in \mathbb{R}^- \mid \text{there exists } y \in \boldsymbol{y} \cap \mathbb{Q}_{even} \text{ with } \exp(y \cdot \log(\text{-}x)) \in \boldsymbol{z}\}\\
&\subseteq \{x \in \mathbb{R}^- \mid \text{there exists } y \in \boldsymbol{y}_{even} \text{ with } \exp(y \cdot \log(\text{-}x)) \in \boldsymbol{z}\}\\
&= \text{-}\{x \in \mathbb{R}^+ \mid \text{there exists } y \in \boldsymbol{y}_{even} \text{ with } \exp(y \cdot \log(x)) \in \boldsymbol{z}\}\\
&= \text{-pow1}_1^-(\boldsymbol{y}_{even}, \boldsymbol{z}),\\
A_4 &= \{x \in \mathbb{R}^- \mid \text{there exists } y \in \boldsymbol{y} \cap \mathbb{Q}_{odd} \text{ with -}\exp(y \cdot \log(\text{-}x)) \in \boldsymbol{z}\}\\
&\subseteq \{x \in \mathbb{R}^- \mid \text{there exists } y \in \boldsymbol{y}_{odd} \text{ with -}\exp(y \cdot \log(\text{-}x)) \in \boldsymbol{z}\}\\
&= \text{-}\{x \in \mathbb{R}^+ \mid \text{there exists } y \in \boldsymbol{y}_{odd} \text{ with } \exp(y \cdot \log(x)) \in \text{-}\boldsymbol{z}\}\\
&= \text{-pow1}_1^-(\boldsymbol{y}_{odd}, \text{-}\boldsymbol{z}).
\end{aligned}$$

Regarding the last two sets, $A_3$ and $A_4$, these are not only subsets of respective inverse operations of pow1, but actually are equal, which follows from continuity of pow1 and the fact that $\mathbb{Q}_{even}$ as well as $\mathbb{Q}_{odd}$ are dense in the set of real numbers, cf. the proof of Lemma 3.9. For the other reverse operation, it is obtained

$$\begin{aligned}
B_1 &= \{y \in \mathbb{R} \mid \text{there exists } x \in \boldsymbol{x} \cap \mathbb{R}^+ \text{ with } \exp(y \cdot \log x) \in \boldsymbol{z}\} \\
&= \mathrm{pow1}_2^-(\boldsymbol{x}, \boldsymbol{z}), \\
B_2 &= \{y \in \mathbb{R}^+ \mid 0 \in \boldsymbol{x} \text{ and } 0 \in \boldsymbol{z}\} \\
&= \begin{cases} \mathbb{R}^+ & \text{if } 0 \in \boldsymbol{x} \text{ and } 0 \in \boldsymbol{z}, \\ \emptyset & \text{otherwise,} \end{cases} \\
B_3 &= \{y \in \mathbb{Q}_{even} \mid \text{there exists } x \in \boldsymbol{x} \cap \mathbb{R}^- \text{ with } \exp(y \cdot \log(-x)) \in \boldsymbol{z}\} \\
&= \{y \in \mathbb{Q}_{even} \mid \text{there exists } x \in -\boldsymbol{x} \cap \mathbb{R}^+ \text{ with } \exp(y \cdot \log(x)) \in \boldsymbol{z}\} \\
&= \mathrm{pow1}_2^-(-\boldsymbol{x}, \boldsymbol{z}) \cap \mathbb{Q}_{even}, \\
B_4 &= \{y \in \mathbb{Q}_{odd} \mid \text{there exists } x \in \boldsymbol{x} \cap \mathbb{R}^- \text{ with } -\exp(y \cdot \log(-x)) \in \boldsymbol{z}\} \\
&= \{y \in \mathbb{Q}_{odd} \mid \text{there exists } x \in -\boldsymbol{x} \cap \mathbb{R}^+ \text{ with } \exp(y \cdot \log(x)) \in -\boldsymbol{z}\} \\
&= \mathrm{pow1}_2^-(-\boldsymbol{x}, -\boldsymbol{z}) \cap \mathbb{Q}_{odd}.
\end{aligned}$$

□

COROLLARY. *From equations 3.5 and 3.6 reverse interval operations of* pow3 *can be given,*

$$\begin{aligned}
\mathrm{pow3}_1^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \mathrm{hull}(&\\
&\mathrm{pow1}_1^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \\
&\begin{cases} \{0\} & \text{if } 0 \in \boldsymbol{x} \text{ and } \boldsymbol{y} \cap \mathbb{R}^+ \neq \emptyset \text{ and } 0 \in \boldsymbol{z} \\ \emptyset & \text{otherwise} \end{cases}, \\
&-\mathrm{pow1}_1^-(-\boldsymbol{x}, \boldsymbol{y}_{even}, \boldsymbol{z}), \\
&-\mathrm{pow1}_1^-(-\boldsymbol{x}, \boldsymbol{y}_{odd}, -\boldsymbol{z})), \\
\mathrm{pow3}_2^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \mathrm{hull}(&\\
&\mathrm{pow1}_2^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \\
&\begin{cases} \boldsymbol{y} \cap ]0, +\infty[ & \text{if } 0 \in \boldsymbol{x} \text{ and } 0 \in \boldsymbol{z} \\ \emptyset & \text{otherwise} \end{cases}, \\
&\mathrm{hull}(\mathrm{pow1}_2^-(-\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \cap \mathbb{Q}_{even}), \\
&\mathrm{hull}(\mathrm{pow1}_2^-(-\boldsymbol{x}, \boldsymbol{y}, -\boldsymbol{z}) \cap \mathbb{Q}_{odd})).
\end{aligned}$$

*Intersections with* $\mathbb{Q}_{even}$ *and* $\mathbb{Q}_{odd}$ *are trivial in a floating-point context* $\mathbb{F}$ *with base 2. For every interval* $\boldsymbol{y} \in \overline{\mathbb{IF}}$ *it holds* $\boldsymbol{y}_{even} = \boldsymbol{y}_{odd} = \boldsymbol{y}$*, if* $\underline{y} < \overline{y}$ *or* $\boldsymbol{y} \subset \mathbb{Z}$*; and* $\boldsymbol{y}_{even} = \boldsymbol{y}_{odd} = \emptyset$ *otherwise.*

# Chapter 4

# Conclusion

The purpose of this thesis is to find a general power function for use in interval arithmetic, particularly with regard to the upcoming interval arithmetic standard which is being developed by the IEEE working group P1788.

Apparently, in the history of mathematics exponentiation has sometimes been used more or less as an abbreviatory notation and several definitions from different backgrounds, i.e., algebra and analysis, have been combined in people's minds. For the set of real numbers these definitions agree on many points, but not on all. Points at issue are treated differently depending on the context, which poses problems searching for a common standard.

In my opinion, the general power function ideally is represented by the complex power function, because of its analytic properties. In this thesis a restriction of the complex power function, pow2, is developed, which comprises all results of the complex function that are real numbers. Besides, all cases of real exponentiation that are commonly accepted in mathematics are included by pow2.

In the process a nice result has come with the study of this function: Further restriction of permitted exponentiation to the domain of positive bases induces a more basic power function pow1, which has "nicer" analytical properties, and which can be applied to many practical problems. Although one should not mistake it as a *general* power function, this variant is supposed to be included by an arithmetic standard.

One contentious issue with the definition of general exponentiation is the assignment of real result to powers with negative base and rational exponent. I do not mean to recommend this as the standard behavior of a general power function, but this possibility is looked at with a more complicated power function pow3 in this work. An implementation of this difficult variant is surprisingly easy.

The interval extension and reverse operations are presented for each of the three variants, and a reference implementation in INTLAB shows that the results can be practically implemented. As the interval mappings related to pow2 and to each variant's reverse operations are quite complicated, I expect great benefits from providing them in an interval arithmetic library. The functions make easily usable, comprehensive tools.

The introduction of reverse operations is still under discussion within the P1788 group. The admission of these operations would clear the way for a use of interval arithmetic in constraint programming and equation solving. The results presented in this thesis prove, that, in contrast to addition and subtraction, there exist elementary arithmetic operations that actually have very complicated reverse operations and are difficult to resolve in equations. This underlines the necessity of reverse operations.

One unresolved problem which arose during my work is to find a good compromise between speed and accuracy during the computation of directly rounded powers in floating-point. Particularly, giving *exact* directly rounded results is still an open problem. Known publications on the topic deal with rounding to the nearest number, and cases with directed rounding are still being worked on. Apart from that, the presented solutions focus on completeness and simplicity, so there certainly are other, undiscovered improvement opportunities which permit future research. In particular, optimizations of interval mappings beyond pow1 have intentionally be omitted with regard to clearness of the results.

Finally, I encourage interested readers to find possible applications for the exceptional power function pow3 or its reverse operations, which may have to do with $\text{pow2}_1^-$.

# Acknowledgments

It is a pleasure to thank those who have supported me, and have made this thesis possible. First of all, I would like to thank my supervisors Professor Wolff von Gudenberg and Marco Nehmeier, who so wisely have given me the opportunity to work on such an exciting and relevant topic and have left me plenty of freedom to work on it. Their expertise, attention to detail and careful guidance have added a lot to my work, and my professor's lectures definitely have made a difference in my entire studies of computer science.

Dan Zuras, chair of the IEEE-754 Revision's committee, has given me a good overview of particular parts of the current stage of development in P1788, and has extensively answered my questions on the development of IEEE-754.

In my studies of mathematics, Professor Theo Grundhöfer and Professor Jörn Steuding have uniquely inspired my enthusiasm on the subject and have raised my awareness of mathematical logic and analysis.

Billa Ohlerich has lovingly taught me English for so many years, I owe her much of my proficiency in the language. Additionally, it was her who brought me into contact with Mick Horton, who helped me a lot proofreading the thesis.

The authors of the Wikipedia Encyclopedia article "Exponentiation" have gathered a very nice collection of sources on the topic, especially regarding the debate on the term $0^0$. They notably accelerated my literature research.

Last but not least, I would like to thank my family for the support they provided me through my entire life. In particular I acknowledge my beloved wife, Katja, whose love and encouragement have helped me to finish this thesis.

Thank you.

# Appendix A

# Algorithms

**Input:** $x, n \in \mathbb{N}$
**Output:** $y = x^n$

```
 1: y ← undefined
 2: X ← x
 3: N ← n
 4: while N ≠ 0 do
 5:   if N is even then
 6:     X ← X∘X
 7:     N ← N/2
 8:   else {N is odd}
 9:     if y is undefined then
10:       y ← X
11:     else
12:       y ← y∘X
13:     end if
14:     N ← N − 1
15:   end if
16: end while
```

ALGORITHM A.1. Log-time computation of positive integral powers induced by Lemma 2.4. Base $x$ may be of any type that allows for associative multiplication, which is used in lines 6, 12. If $\circ$ was an additive operator instead, this algorithm would compute the $n$-th multiple of $x$.

**Input:** $\boldsymbol{x} = [\underline{x}, \overline{x}] \in \overline{\mathbb{IF}}$ with $0 < \underline{x}$, $\boldsymbol{y} = [\underline{y}, \overline{y}] \in \overline{\mathbb{IF}}$
**Output:** $\boldsymbol{z} = [\underline{z}, \overline{z}] = \mathrm{pow1}_{\mathbb{F}}(\boldsymbol{x}, \boldsymbol{y}) \in \overline{\mathbb{IF}}$

**if** $\underline{x} < 1$ or $0 \le \underline{y}$ or $\overline{y} \le 0$ **then**
  $\underline{l} \leftarrow \triangledown \log \underline{x}$
**end if**
**if** $1 < \overline{x}$ or $0 \le \underline{y}$ or $\overline{y} \le 0$ **then**
  $\overline{l} \leftarrow \vartriangle \log \overline{x}$
**end if**
**if** $0 \le \underline{y}$ **then**
  **if** $\overline{x} \le 1$ **then**
    $[\underline{m}, \overline{m}] \leftarrow [\triangledown(\overline{y} \cdot \underline{l}), \vartriangle(\underline{y} \cdot \overline{l})]$
  **else if** $1 \le \underline{x}$ **then**
    $[\underline{m}, \overline{m}] \leftarrow [\triangledown(\underline{y} \cdot \underline{l}), \triangledown(\overline{y} \cdot \overline{l})]$
  **else** $\{\underline{x} < 1 < \overline{x}\}$
    $[\underline{m}, \overline{m}] \leftarrow [\triangledown(\overline{y} \cdot \underline{l}), \vartriangle(\overline{y} \cdot \overline{l})]$
  **end if**
**else if** $\overline{y} \le 0$ **then**
  **if** $\overline{x} \le 1$ **then**
    $[\underline{m}, \overline{m}] \leftarrow [\triangledown(\overline{y} \cdot \overline{l}), \vartriangle(\underline{y} \cdot \underline{l})]$
  **else if** $1 \le \underline{x}$ **then**
    $[\underline{m}, \overline{m}] \leftarrow [\triangledown(\underline{y} \cdot \overline{l}), \triangledown(\overline{y} \cdot \underline{l})]$
  **else** $\{\underline{x} < 1 < \overline{x}\}$
    $[\underline{m}, \overline{m}] \leftarrow [\triangledown(\underline{y} \cdot \overline{l}), \vartriangle(\underline{y} \cdot \underline{l})]$
  **end if**
**else** $\{\underline{y} < 0 < \overline{y}\}$
  **if** $\overline{x} \le 1$ **then**
    $[\underline{m}, \overline{m}] \leftarrow [\triangledown(\overline{y} \cdot \underline{l}), \vartriangle(\underline{y} \cdot \underline{l})]$
  **else if** $1 \le \underline{x}$ **then**
    $[\underline{m}, \overline{m}] \leftarrow [\triangledown(\underline{y} \cdot \overline{l}), \vartriangle(\overline{y} \cdot \overline{l})]$
  **else** $\{\underline{x} < 1 < \overline{x}\}$
    $[\underline{m}, \overline{m}] \leftarrow [\min\{\triangledown(\overline{y} \cdot \underline{l}), \triangledown(\underline{y} \cdot \overline{l})\}, \max\{\vartriangle(\underline{y} \cdot \underline{l}), \vartriangle(\overline{y} \cdot \overline{l})\}]$
  **end if**
**end if**
$[\underline{z}, \overline{z}] \leftarrow [\triangledown \exp \underline{m}, \vartriangle \exp \overline{m}]$

ALGORITHM A.2. Computation of $\mathrm{pow1}_{\mathbb{F}}$. Operators $\vartriangle$ and $\triangledown$ denote upward/downward rounding to the next higher/lower number of $\mathbb{F}$. It is assumed that logarithm, multiplication and exponential function are provided by the floating-point context with directed rounding.

**Input:** $\boldsymbol{x}$, $\boldsymbol{y}$
**Output:** $\boldsymbol{z} = \text{pow2}(\boldsymbol{x}, \boldsymbol{y})$

$\boldsymbol{x}^- \leftarrow \boldsymbol{x} \cap ]\text{-}\infty, 0]$
$\boldsymbol{z}^+ \leftarrow \text{pow1}(\boldsymbol{x}, \boldsymbol{y})$ with Table 3.3
**if** $\boldsymbol{y}$ contains no integer **then**
  $\boldsymbol{z}^- \leftarrow \emptyset$
**else if** $\boldsymbol{y}$ contains only one integer **then**
  $\boldsymbol{z}^- \leftarrow \text{pow2}(\boldsymbol{x}^-, \boldsymbol{y})$ with Table 3.5
**else** $\{\boldsymbol{y}$ contains at least two integers$\}$
  $\boldsymbol{z}^- \leftarrow \text{pow2}(\boldsymbol{x}^-, \boldsymbol{y})$ with Table 3.4
**end if**
$\boldsymbol{z} \leftarrow \begin{cases} \{0\} & \text{if } \boldsymbol{y} \cap \mathbb{R}^+ \neq \emptyset \text{ and } 0 \in \boldsymbol{x}, \\ \emptyset & \text{otherwise} \end{cases}$
$\boldsymbol{z} \leftarrow \text{hull}(\boldsymbol{z} \cup \boldsymbol{z}^- \cup \boldsymbol{z}^+)$

Algorithm A.3. Computation of pow2. Cases where $\boldsymbol{x}$ or $\boldsymbol{y}$ are unbound or $0 \in \boldsymbol{x}$ are not supported by the given tables and demand computation of respective limit values.

**Input:** $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$
**Output:** $\boldsymbol{h} = \text{pow2}_1^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \text{hull}(\boldsymbol{x} \cap \text{pow2}_1^-(\boldsymbol{y}, \boldsymbol{z}))$

$\boldsymbol{h}^+ \leftarrow \text{pow1}_1^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$  $\{= \text{hull}(\boldsymbol{x} \cap \text{pow1}_1^-(\boldsymbol{y}, \boldsymbol{z}))$ with Table B.1$\}$
$\boldsymbol{x}^- \leftarrow \boldsymbol{x} \cap ]\text{-}\infty, 0]$
$\boldsymbol{h}^- \leftarrow \text{hull}(\boldsymbol{x}^- \cap \text{pow2}_1^-(\boldsymbol{y}, \boldsymbol{z}))$ with Table B.3
$\boldsymbol{h} \leftarrow \text{hull}(\boldsymbol{h}^- \cup \boldsymbol{h}^+)$

Algorithm A.4. Computation of $\text{pow2}_1^-$. Cases where $\boldsymbol{y}$ or $\boldsymbol{z}$ are unbound are not supported by the given tables and demand computation of respective limit values.

**Input:** $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$
**Output:** $\boldsymbol{h} = \text{pow2}_2^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \text{hull}(\boldsymbol{y} \cap \text{pow2}_2^-(\boldsymbol{x}, \boldsymbol{z}))$

$\boldsymbol{h}^+ \leftarrow \text{pow1}_2^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$  $\{= \text{hull}(\boldsymbol{y} \cap \text{pow1}_2^-(\boldsymbol{x}, \boldsymbol{z}))$ with Table B.2$\}$
$\boldsymbol{x}^- \leftarrow \boldsymbol{x} \cap ]\text{-}\infty, 0]$
$\boldsymbol{h}^- \leftarrow \text{hull}(\boldsymbol{y} \cap \text{pow2}_2^-(\boldsymbol{x}^-, \boldsymbol{z}))$ with Table B.4
$\boldsymbol{h} \leftarrow \text{hull}(\boldsymbol{h}^- \cup \boldsymbol{h}^+)$

Algorithm A.5. Computation of $\text{pow2}_2^-$. Cases where $\boldsymbol{x}$ or $\boldsymbol{z}$ are unbound are not supported by the given tables and demand computation of respective limit values.

**Input:** $\boldsymbol{x} = [\underline{x}, \overline{x}]$ with $\overline{x} \le 0$, $\boldsymbol{y} = [\underline{y}, \overline{y}]$ with $\underline{y} > 0$, $\boldsymbol{z} = [\underline{z}, \overline{z}]$ with $\underline{z} > 1$

**Output:** $[\underline{r}, \overline{r}] = \mathrm{pow2}_1^-(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{x} \cap \bigcup_{\substack{n \in \boldsymbol{y} \\ n \text{ even}}} [-|\overline{z}|^{1/n}, -|\underline{z}|^{1/n}]$

$[\underline{h}, \overline{h}] \leftarrow [-|\overline{z}|^{1/lee}, -|\underline{z}|^{1/gee}]$ {enclosure of union of intervals}
$[\underline{r}, \overline{r}] \leftarrow [\underline{x}, \overline{x}] \cap [\underline{h}, \overline{h}]$ {enclosure of result}
**if** $\underline{h} < \underline{x} < \overline{h}$ **then**
  {optimize left boundary of result}
  $a \leftarrow -\log_{|\underline{x}|} |\underline{z}|$
  $b \leftarrow \min\{y \in \mathbb{Z} \mid y \text{ even and } y \ge a\}$
  $c \leftarrow -|\overline{z}|^{1/b}$
  $\underline{r} \leftarrow \max\{\underline{r}, c\}$
**end if**
**if** $\underline{h} < \overline{x} < \overline{h}$ **then**
  {optimize right boundary of result}
  $a \leftarrow -\log_{|\overline{x}|} |\overline{z}|$
  $b \leftarrow \max\{y \in \mathbb{Z} \mid y \text{ even and } y \le a\}$
  $c \leftarrow -|\underline{z}|^{1/b}$
  $\overline{r} \leftarrow \min\{\overline{r}, c\}$
**end if**
{if $\underline{r} > \overline{r}$, the result is the empty set}

ALGORITHM A.6. Computation of $\mathrm{pow2}_1^-$ in cases where $\overline{x} \le 0$, $\underline{y} > 0$ and $\underline{z} > 1$.

# Appendix B

# Reverse Operation Tables

Table B.1. $\mathrm{pow1}_1^-(\boldsymbol{y},\boldsymbol{z})$ with bounded, non-empty intervals $\boldsymbol{y} = [\underline{y},\overline{y}]$ and $\boldsymbol{z} = [\underline{z},\overline{z}]$, where $0 < \overline{z}$.

| | $\boldsymbol{y} \circledcirc [0,0]$ | | | | | |
|---|---|---|---|---|---|---|
| $\boldsymbol{z} \circledcirc [0,1]$ | beforeP | equalP | finishedBy | containsP | startedBy | afterP |
| overlaps/starts | $[\overline{z}^{1/\underline{y}}, +\infty[$ | $\emptyset$ | $[\overline{z}^{1/\underline{y}}, +\infty[$ | $]0, \overline{z}^{1/\overline{y}}] \cup [\overline{z}^{1/\underline{y}}, +\infty[$ | $]0, \overline{z}^{1/\overline{y}}]$ | $]0, \overline{z}^{1/\overline{y}}]$ |
| containedByP | $[\overline{z}^{1/\underline{y}}, \underline{z}^{1/\overline{y}}]$ | $\emptyset$ | $[\overline{z}^{1/\underline{y}}, +\infty[$ | $]0, \overline{z}^{1/\overline{y}}] \cup [\overline{z}^{1/\underline{y}}, +\infty[$ | $]0, \overline{z}^{1/\overline{y}}]$ | $[\underline{z}^{1/\underline{y}}, \overline{z}^{1/\overline{y}}]$ |
| finishes | $[1, \underline{z}^{1/\overline{y}}]$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $[\underline{z}^{1/\underline{y}}, 1]$ |
| equalP/finishedBy | $[1, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, 1]$ |
| containsP/startedBy | $[\overline{z}^{1/\overline{y}}, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, \overline{z}^{1/\underline{y}}]$ |
| overlappedBy | $[\overline{z}^{1/\overline{y}}, \underline{z}^{1/\overline{y}}]$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $[\underline{z}^{1/\underline{y}}, \overline{z}^{1/\underline{y}}]$ |
| metBy | $[\overline{z}^{1/\overline{y}}, 1]$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $]0, +\infty[$ | $[1, \overline{z}^{1/\underline{y}}]$ |
| afterP | $[\overline{z}^{1/\overline{y}}, \underline{z}^{1/\underline{y}}]$ | $\emptyset$ | $]0, \underline{z}^{1/\underline{y}}]$ | $]0, \underline{z}^{1/\underline{y}}] \cup [\underline{z}^{1/\overline{y}}, +\infty[$ | $[\underline{z}^{1/\overline{y}}, +\infty[$ | $[\underline{z}^{1/\overline{y}}, \overline{z}^{1/\underline{y}}]$ |

*Note:* For $\overline{z} \le 0$ the result is $\emptyset$. Results for unbounded intervals can easily be obtained via limit values: For example, assuming $\boldsymbol{y}$ afterP $[0,0]$ and $\boldsymbol{z}$ afterP $[0,1]$ with $\overline{z} = +\infty$ leads to $\mathrm{pow1}_1^-(\boldsymbol{y},\boldsymbol{z}) = [\underline{z}^{1/\overline{y}}, \overline{z}^{1/\underline{y}}]$ in the bounded case. For the unbounded case $\mathrm{pow1}_1^-(\boldsymbol{y},\boldsymbol{z}) = [\underline{z}^{1/\overline{y}}, \lim_{\overline{z}\to+\infty} \overline{z}^{1/\underline{y}}[ = [\underline{z}^{1/\overline{y}}, +\infty[$ is obtained. An interval boundary that comes from a limit value is not part of the result. Any other cases can be handled analogically and limits are unique, cf. Table 2.3. Sketches of overlapping states are given on a detachable card as an appendix.

Table B.2. $\text{pow1}_2^-(\boldsymbol{x},\boldsymbol{z})$ with bounded, non-empty intervals $\boldsymbol{x} = [\underline{x},\overline{x}]$ and $\boldsymbol{z} = [\underline{z},\overline{z}]$, where $0 < \overline{x}$ and $0 < \overline{z}$.

| | $\boldsymbol{x} \circledcirc [0,1]$ | | | | |
|---|---|---|---|---|---|
| $\boldsymbol{z} \circledcirc [0,1]$ | overlaps/starts | containedByP | finishes | equalP/finishedBy | … |
| overlaps/starts | $]0,+\infty[$ | $[\log_{\underline{x}} \overline{z},+\infty[$ | $\begin{cases} [\log_{\underline{x}} \overline{z},+\infty[ & \text{if } \underline{x} < 1, \\ \emptyset & \text{if } \underline{x} = 1. \end{cases}$ | $]0,+\infty[$ | |
| containedByP | $]0,\log_{\overline{x}} \underline{z}]$ | $[\log_{\underline{x}} \overline{z},\log_{\overline{x}} \underline{z}]$ | $\begin{cases} [\log_{\underline{x}} \overline{z},+\infty[ & \text{if } \underline{x} < 1, \\ \emptyset & \text{if } \underline{x} = 1. \end{cases}$ | $]0,+\infty[$ | |
| finishes | $[0,\log_{\overline{x}} \underline{z}]$ | $[0,\log_{\overline{x}} \underline{z}]$ | $]-\infty,+\infty[$ | $]-\infty,+\infty[$ | |
| equalP/finishedBy | $[0,+\infty[$ | $[0,+\infty[$ | $]-\infty,+\infty[$ | $]-\infty,+\infty[$ | |
| containsP/startedBy | $[\log_{\overline{x}} \overline{z},+\infty[$ | $[\log_{\overline{x}} \overline{z},+\infty[$ | $]-\infty,+\infty[$ | $]-\infty,+\infty[$ | |
| overlappedBy | $[\log_{\overline{x}} \overline{z},\log_{\overline{x}} \underline{z}]$ | $[\log_{\overline{x}} \overline{z},\log_{\overline{x}} \underline{z}]$ | $]-\infty,+\infty[$ | $]-\infty,+\infty[$ | |
| metBy | $[\log_{\overline{x}} \overline{z},0]$ | $[\log_{\overline{x}} \overline{z},0]$ | $]-\infty,+\infty[$ | $]-\infty,+\infty[$ | |
| afterP | $[\log_{\overline{x}} \overline{z},0[$ | $[\log_{\overline{x}} \overline{z},\log_{\underline{x}} \underline{z}]$ | $\begin{cases} [-\infty,\log_{\underline{x}} \underline{z}] & \text{if } \underline{x} < 1, \\ \emptyset & \text{if } \underline{x} = 1. \end{cases}$ | $]-\infty,0[$ | |

*Note:* For $\overline{x} \leq 0$ or $\overline{z} \leq 0$ the result is $\emptyset$. Results for unbounded intervals can easily be obtained via limit values: For example, assuming $\boldsymbol{x}$ starts $[0,1]$ and $\boldsymbol{z}$ afterP $[0,1]$ with $\overline{z} = +\infty$ leads to $\text{pow1}_2^-(\boldsymbol{x},\boldsymbol{z}) = [\log_{\overline{x}} \overline{z},0[$ in the bounded case. For the unbounded case $\text{pow1}_2^-(\boldsymbol{x},\boldsymbol{z}) = ]\lim_{\overline{z}\to+\infty} \log_{\overline{x}} \overline{z},0[ = ]-\infty,0[$ is obtained. An interval boundary that comes from a limit value is not part of the result. Any other cases can be handled analogically and limits are unique. Sketches of overlapping states are given on a detachable card as an appendix.

TABLE B.2 *(CONTINUED)*

| | $\boldsymbol{x}\circledcirc[0,1]$ | | | |
|---|---|---|---|---|
| $\boldsymbol{z}\circledcirc[0,1]$ | containsP/startedBy | overlappedBy | metBy | afterP |
| overlaps/starts | $]-\infty, \log_{\overline{x}} \overline{z}] \cup ]0, +\infty[$ | $]-\infty, \log_{\overline{x}} \overline{z}] \cup [\log_{\underline{x}} \overline{z}, +\infty[$ | $]-\infty, \log_{\overline{x}} \overline{z}]$ | $]-\infty, \log_{\overline{x}} \overline{z}]$ |
| containedByP | $]-\infty, \log_{\overline{x}} \overline{z}] \cup ]0, +\infty[$ | $]-\infty, \log_{\overline{x}} \overline{z}] \cup [\log_{\underline{x}} \overline{z}, +\infty[$ | $]-\infty, \log_{\overline{x}} \overline{z}]$ | $[\log_{\underline{x}} \underline{z}, \log_{\overline{x}} \overline{z}]$ |
| finishes | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $[\log_{\underline{x}} \underline{z}, 0]$ |
| equalP/finishedBy | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $]-\infty, 0]$ |
| containsP/startedBy | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $]-\infty, \log_{\underline{x}} \overline{z}]$ |
| overlappedBy | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $[\log_{\underline{x}} \underline{z}, \log_{\underline{x}} \overline{z}]$ |
| metBy | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $]-\infty, +\infty[$ | $[0, \log_{\underline{x}} \overline{z}]$ |
| afterP | $]-\infty, 0[ \cup [\log_{\overline{x}} \underline{z}, +\infty[$ | $]-\infty, \log_{\underline{x}} \underline{z}] \cup [\log_{\overline{x}} \underline{z}, +\infty[$ | $[\log_{\overline{x}} \underline{z}, +\infty[$ | $[\log_{\overline{x}} \underline{z}, \log_{\underline{x}} \overline{z}]$ |

TABLE B.3. $\mathrm{pow2}_1^-(\boldsymbol{y},\boldsymbol{z})$ with bounded, non-empty intervals $\boldsymbol{y} = [\underline{y},\overline{y}]$ and $\boldsymbol{z} = [\underline{z},\overline{z}]$.

| $\boldsymbol{z}$ ⊚ [-1, 1] | $\boldsymbol{y}$ ⊚ [0, 0] beforeP | equalP | afterP |
|---|---|---|---|
| Ⓐ/Ⓔ | $\bigcup_{n\in\boldsymbol{y},\ n \text{ odd}} [-\lvert\overline{z}\rvert^{1/n}, -\lvert\underline{z}\rvert^{1/n}]$ | ∅ | $\bigcup_{n\in\boldsymbol{y},\ n \text{ odd}} [-\lvert\underline{z}\rvert^{1/n}, -\lvert\overline{z}\rvert^{1/n}]$ |
| Ⓑ | $[-1, -\lvert\underline{z}\rvert^{1/goe}]$ | ∅ | $[-\lvert\underline{z}\rvert^{1/loe}, -1]$ |
| Ⓒ | $[-\lvert\overline{z}\rvert^{1/goe}, -\lvert\underline{z}\rvert^{1/goe}]$ | ∅ | $[-\lvert\underline{z}\rvert^{1/loe}, -\lvert\overline{z}\rvert^{1/loe}]$ |
| Ⓓ | $[-\lvert\overline{z}\rvert^{1/goe}, -1]$ | ∅ | $[-1, -\lvert\overline{z}\rvert^{1/loe}]$ |
| Ⓕ | $]-\infty, -\lvert\underline{z}\rvert^{1/loe}]$ | ∅ | $[-\lvert\underline{z}\rvert^{1/goe}, 0] \cup \{0\}$ |
| Ⓖ | ∅ | ∅ | $\{0\}$ |
| Ⓗ | $]-\infty, -1^{1/goe}]$ | ∅ | $[-1^{1/loe}, 0] \cup \{0\}$ |
| Ⓘ | $]-\infty, -\lvert\underline{z}\rvert^{1/goe}]$ | ∅ | $[-\lvert\underline{z}\rvert^{1/loe}, 0] \cup \{0\}$ |
| Ⓙ | $]-\infty, -\lvert\underline{z}\rvert^{1/goe}] \cup ]-\infty, -\overline{z}^{1/lee}]$ | ∅ | $[-\lvert\underline{z}\rvert^{1/loe}, 0] \cup [-\overline{z}^{1/gee}, 0] \cup \{0\}$ |
| Ⓚ | $]-\infty, -\lvert\underline{z}\rvert^{1/goe}] \cup ]-\infty, -1^{1/lee}]$ | $]-\infty, 0[$ | $[-\lvert\underline{z}\rvert^{1/loe}, 0] \cup [-1^{1/lee}, 0] \cup \{0\}$ |
| Ⓛ | $]-\infty, -\lvert\underline{z}\rvert^{1/goe}] \cup ]-\infty, -\overline{z}^{1/gee}]$ | $]-\infty, 0[$ | $[-\lvert\underline{z}\rvert^{1/loe}, 0] \cup [-\overline{z}^{1/lee}, 0] \cup \{0\}$ |
| Ⓜ | $]-\infty, -1^{1/goe}] \cup ]-\infty, -\overline{z}^{1/lee}]$ | ∅ | $[-1^{1/loe}, 0] \cup ]-\overline{z}^{1/gee}, 0] \cup \{0\}$ |
| Ⓝ | $]-\infty, -1^{1/goe}] \cup ]-\infty, -1^{1/lee}]$ | $]-\infty, 0[$ | $[-1^{1/loe}, 0] \cup [-1^{1/gee}, 0] \cup \{0\}$ |
| Ⓞ | $]-\infty, -1^{1/goe}] \cup ]-\infty, -\overline{z}^{1/gee}]$ | $]-\infty, 0[$ | $[-1^{1/loe}, 0] \cup [-\overline{z}^{1/lee}, 0] \cup \{0\}$ |
| Ⓟ | $]-\infty, -\lvert\underline{z}\rvert^{1/loe}] \cup ]-\infty, -\overline{z}^{1/lee}]$ | ∅ | $[-\lvert\underline{z}\rvert^{1/goe}, 0] \cup ]-\overline{z}^{1/gee}, 0] \cup \{0\}$ |
| Ⓠ | $]-\infty, -\lvert\underline{z}\rvert^{1/loe}] \cup ]-\infty, -1^{1/lee}]$ | $]-\infty, 0[$ | $[-\lvert\underline{z}\rvert^{1/goe}, 0] \cup [-1^{1/gee}, 0] \cup \{0\}$ |
| Ⓡ | $]-\infty, -\lvert\underline{z}\rvert^{1/loe}] \cup ]-\infty, -\overline{z}^{1/gee}]$ | $]-\infty, 0[$ | $[-\lvert\underline{z}\rvert^{1/goe}, 0] \cup [-\overline{z}^{1/lee}, 0] \cup \{0\}$ |
| Ⓢ | $]-\infty, -\overline{z}^{1/lee}]$ | ∅ | $[-\overline{z}^{1/gee}, 0] \cup \{0\}$ |
| Ⓣ | $]-\infty, -1^{1/lee}]$ | $]-\infty, 0[$ | $[-1^{1/gee}, 0] \cup \{0\}$ |
| Ⓤ | $]-\infty, -\overline{z}^{1/gee}]$ | $]-\infty, 0[$ | $[-\overline{z}^{1/lee}, 0] \cup \{0\}$ |
| Ⓥ/Ⓩ | $\bigcup_{n\in\boldsymbol{y},\ n \text{ even}} [-\underline{z}^{1/n}, -\overline{z}^{1/n}]$ | ∅ | $\bigcup_{n\in\boldsymbol{y},\ n \text{ even}} [-\overline{z}^{1/n}, -\underline{z}^{1/n}]$ |
| Ⓦ | $[-\underline{z}^{1/gee}, -1]$ | $]-\infty, 0[$ | $[-1, -\underline{z}^{1/lee}]$ |
| Ⓧ | $[-\underline{z}^{1/gee}, -\overline{z}^{1/gee}]$ | $]-\infty, 0[$ | $[-\overline{z}^{1/lee}, -\underline{z}^{1/lee}]$ |
| Ⓨ | $[-1, -\overline{z}^{1/gee}]$ | $]-\infty, 0[$ | $[-\overline{z}^{1/lee}, -1]$ |

*Note:* Within the interval $[\underline{y},\overline{y}]$ it is denoted *lee*: lowest even exponent, *gee*: greatest even exponent, *loe*: lowest odd exponent, *goe*: greatest odd exponent, all four numbers might or might not exist. If one such number does not exist, intervals using it are considered to be empty. Missing states of $\boldsymbol{y}$ ⊚ [0, 0] can be put together, e.g., finishedBy = beforeP ∪ equalP. Positive results need to be looked up in Table B.1. Sketches of overlapping states are given on a detachable card as an appendix.

TABLE B.4. $\text{pow2}_2^-(\boldsymbol{x},\boldsymbol{z})$ with bounded, non-empty intervals $\boldsymbol{x} = [\underline{x},\overline{x}]$ and $\boldsymbol{z} = [\underline{z},\overline{z}]$ with $\underline{x} \leq 0$.

| | $\boldsymbol{x} \circledcirc [\text{-}1,0]$ | |
|---|---|---|
| $\boldsymbol{z} \circledcirc\!\circ [\text{-}1,1]$ | beforeP | … |
| Ⓐ | $\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\overline{z}\rvert, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert]$ | |
| Ⓑ | $\mathbb{Z}_1 \cap ]0, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert]$ | |
| Ⓒ | $\mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\overline{z}\rvert, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert]$ | |
| Ⓓ | $\begin{cases} \mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\overline{z}\rvert, 0[ & \text{if } \underline{z} < \overline{z} \\ \emptyset & \text{if } \underline{z} = \overline{z} \end{cases}$ | |
| Ⓔ | $\mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\overline{z}\rvert, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert]$ | |
| Ⓕ | $\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert]$ | |
| Ⓖ | $\emptyset$ | |
| Ⓗ | $\mathbb{Z}_1 \cap ]\text{-}\infty, 0[$ | |
| Ⓘ | $\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert]$ | |
| Ⓙ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert]) \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \overline{z}])$ | |
| Ⓚ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert]) \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, 0])$ | |
| Ⓛ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert]) \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \overline{z}])$ | |
| Ⓜ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, 0[) \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \overline{z}])$ | |
| Ⓝ | $\mathbb{Z} \cap ]\text{-}\infty, 0]$ | |
| Ⓞ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, 0[) \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \overline{z}])$ | |
| Ⓟ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert]) \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \overline{z}])$ | |
| Ⓠ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert]) \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, 0])$ | |
| Ⓡ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert]) \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \overline{z}])$ | |
| Ⓢ | $\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \overline{z}]$ | |
| Ⓣ | $\mathbb{Z}_2 \cap ]\text{-}\infty, 0]$ | |
| Ⓤ | $\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \overline{z}]$ | |
| Ⓥ | $\mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \underline{z}, \log_{\lvert\underline{x}\rvert} \overline{z}]$ | |
| Ⓦ | $\begin{cases} \mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \underline{z}, 0] & \text{if } \underline{z} < \overline{z} \\ \emptyset & \text{if } \underline{z} = \overline{z} \end{cases}$ | |
| Ⓧ | $\mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \underline{z}, \log_{\lvert\overline{x}\rvert} \overline{z}]$ | |
| Ⓨ | $\mathbb{Z}_2 \cap [0, \log_{\lvert\overline{x}\rvert} \overline{z}]$ | |
| Ⓩ | $\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \underline{z}, \log_{\lvert\overline{x}\rvert} \overline{z}]$ | |

*Note:* $\mathbb{Z}_1 = \{n \in \mathbb{Z} \mid n \text{ odd}\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z} \mid n \text{ even}\}$. For $0 < \underline{x}$ results need to be looked up in Table B.2. Sketches of overlapping states are given on a detachable card as an appendix.

Table B.4 (*continued*)

| | $\boldsymbol{x}$ ⚭ $[\text{-}1,0]$ | |
|---|---|---|
| $\boldsymbol{z}$ ⚭ↀ $[\text{-}1,1]$ | meets | … |
| Ⓐ | $\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\overline{z}\rvert, +\infty[$ | |
| Ⓑ | $\mathbb{Z}_1 \cap ]0, +\infty[$ | |
| Ⓒ/Ⓗ/Ⓘ | $\mathbb{Z}_1$ | |
| Ⓓ | $\mathbb{Z}_1 \cap ]\text{-}\infty, 0[$ | |
| Ⓔ/Ⓕ | $\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert]$ | |
| Ⓖ | $\emptyset$ | |
| Ⓙ/Ⓜ | $\mathbb{Z}_1 \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \overline{z}])$ | |
| Ⓚ/Ⓛ/Ⓝ/Ⓞ | $\mathbb{Z}$ | |
| Ⓟ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert]) \cup (\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \overline{z}])$ | |
| Ⓠ/Ⓡ | $(\mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert]) \cup \mathbb{Z}_2$ | |
| Ⓢ/Ⓥ | $\mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\underline{x}\rvert} \overline{z}]$ | |
| Ⓣ/Ⓤ/Ⓦ/Ⓧ/Ⓨ | $\mathbb{Z}_2$ | |
| Ⓩ | $\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \underline{z}, +\infty[$ | |

*Note:* $\mathbb{Z}_1 = \{n \in \mathbb{Z} \mid n \text{ odd}\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z} \mid n \text{ even}\}$.

Table B.4 (*continued*)

| | $\boldsymbol{x}$ ⚭ $[\text{-}1,0]$ | |
|---|---|---|
| $\boldsymbol{z}$ ⚭ↀ $[\text{-}1,1]$ | overlaps | … |
| Ⓐ | $\mathbb{Z}_1 \setminus ]\log_{\lvert\overline{x}\rvert} \lvert\overline{z}\rvert, \log_{\lvert\underline{x}\rvert} \lvert\overline{z}\rvert[$ | |
| Ⓑ/Ⓒ/Ⓓ/Ⓗ/Ⓘ | $\mathbb{Z}_1$ | |
| Ⓔ/Ⓕ | $\mathbb{Z}_1 \setminus ]\log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert[$ | |
| Ⓖ | $\emptyset$ | |
| Ⓙ/Ⓜ | $\mathbb{Z}_1 \cup (\mathbb{Z}_2 \setminus ]\log_{\lvert\underline{x}\rvert} \overline{z}, \log_{\lvert\overline{x}\rvert} \overline{z}[)$ | |
| Ⓚ/Ⓛ/Ⓝ/Ⓞ | $\mathbb{Z}$ | |
| Ⓟ | $(\mathbb{Z}_1 \setminus ]\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert[) \cup (\mathbb{Z}_2 \setminus ]\log_{\lvert\underline{x}\rvert} \overline{z}, \log_{\lvert\overline{x}\rvert} \overline{z}[)$ | |
| Ⓠ/Ⓡ | $(\mathbb{Z}_1 \setminus ]\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert[) \cup \mathbb{Z}_2$ | |
| Ⓢ/Ⓥ | $\mathbb{Z}_2 \setminus ]\log_{\lvert\underline{x}\rvert} \overline{z}, \log_{\lvert\overline{x}\rvert} \overline{z}[$ | |
| Ⓣ/Ⓤ/Ⓦ/Ⓧ/Ⓨ | $\mathbb{Z}_2$ | |
| Ⓩ | $\mathbb{Z}_2 \setminus ]\log_{\lvert\underline{x}\rvert} \underline{z}, \log_{\lvert\overline{x}\rvert} \underline{z}[$ | |

*Note:* $\mathbb{Z}_1 = \{n \in \mathbb{Z} \mid n \text{ odd}\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z} \mid n \text{ even}\}$.

Table B.4 *(continued)*

| | $\boldsymbol{x}$⚭[-1,0] | |
|---|---|---|
| $\boldsymbol{z}$⚭⚬[-1,1] | starts | … |
| Ⓐ | $\begin{cases} \mathbb{Z}_1 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \lvert\overline{z}\rvert] & \text{if } \underline{x} < \overline{x} \\ \varnothing & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓑ/Ⓒ/Ⓓ/Ⓗ/Ⓘ | $\mathbb{Z}_1$ | |
| Ⓔ/Ⓕ | $\begin{cases} \mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert, +\infty[ & \text{if } \underline{x} < \overline{x} \\ \varnothing & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓖ | $\varnothing$ | |
| Ⓙ/Ⓜ | $\begin{cases} \mathbb{Z}_1 \cup (\mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \overline{z}, +\infty[) & \text{if } \underline{x} < \overline{x} \\ \mathbb{Z}_1 & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓚ/Ⓛ/Ⓝ/Ⓞ | $\mathbb{Z}$ | |
| Ⓟ | $\begin{cases} (\mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup (\mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \overline{z}, +\infty[) & \text{if } \underline{x} < \overline{x} \\ \varnothing & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓠ/Ⓡ | $\begin{cases} (\mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup \mathbb{Z}_2 & \text{if } \underline{x} < \overline{x} \\ \mathbb{Z}_2 & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓢ/Ⓥ | $\begin{cases} \mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \lvert\overline{z}\rvert, +\infty[ & \text{if } \underline{x} < \overline{x} \\ \varnothing & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓣ/Ⓤ/Ⓦ/Ⓧ/Ⓨ | $\mathbb{Z}_2$ | |
| Ⓩ | $\begin{cases} \mathbb{Z}_2 \cap ]\text{-}\infty, \log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert] & \text{if } \underline{x} < \overline{x} \\ \varnothing & \text{if } \underline{x} = \overline{x} \end{cases}$ | |

*Note:* $\mathbb{Z}_1 = \{n \in \mathbb{Z} \mid n \text{ odd}\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z} \mid n \text{ even}\}$.

TABLE B.4 *(CONTINUED)*

| | $\boldsymbol{x}$ꝏ[-1, 0] | |
|---|---|---|
| $\boldsymbol{z}$ꝏo[-1, 1] | containedByP | … |
| Ⓐ | $\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, \log_{\lvert\overline{x}\rvert} \lvert\overline{z}\rvert]$ | |
| Ⓑ | $\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, 0[$ | |
| Ⓒ | $\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\overline{z}\rvert, \log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert]$ | |
| Ⓓ | $\begin{cases} \mathbb{Z}_1 \cap ]0, \log_{\lvert\underline{x}\rvert} \lvert\overline{z}\rvert] & \text{if } \underline{z} < \overline{z} \\ \emptyset & \text{if } \underline{z} = \overline{z} \end{cases}$ | |
| Ⓔ | $\mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert, \log_{\lvert\underline{x}\rvert} \lvert\overline{z}\rvert]$ | |
| Ⓕ | $\mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert, +\infty[$ | |
| Ⓖ | $\emptyset$ | |
| Ⓗ | $\mathbb{Z}_1 \cap ]0, +\infty[$ | |
| Ⓘ | $\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, +\infty[$ | |
| Ⓙ | $(\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup (\mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \overline{z}, +\infty[)$ | |
| Ⓚ | $(\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup (\mathbb{Z}_2 \cap [0, +\infty[)$ | |
| Ⓛ | $(\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup (\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, +\infty[)$ | |
| Ⓜ | $(\mathbb{Z}_1 \cap ]0, +\infty[) \cup (\mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \overline{z}, +\infty[)$ | |
| Ⓝ | $\mathbb{Z} \cap [0, +\infty[$ | |
| Ⓞ | $(\mathbb{Z}_1 \cap ]0, +\infty[) \cup (\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, +\infty[$ | |
| Ⓟ | $(\mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup (\mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \overline{z}, +\infty[)$ | |
| Ⓠ | $(\mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup (\mathbb{Z}_2 \cap [0, +\infty[)$ | |
| Ⓡ | $(\mathbb{Z}_1 \cap [\log_{\lvert\overline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup (\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, +\infty[)$ | |
| Ⓢ | $\mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \overline{z}, +\infty[$ | |
| Ⓣ | $\mathbb{Z}_2 \cap [0, +\infty[$ | |
| Ⓤ | $\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, +\infty[$ | |
| Ⓥ | $\mathbb{Z}_2 \cap [\log_{\lvert\overline{x}\rvert} \overline{z}, \log_{\lvert\underline{x}\rvert} \underline{z}]$ | |
| Ⓦ | $\begin{cases} \mathbb{Z}_2 \cap [0, \log_{\lvert\underline{x}\rvert} \underline{z}] & \text{if } \underline{z} < \overline{z} \\ \emptyset & \text{if } \underline{z} = \overline{z} \end{cases}$ | |
| Ⓧ | $\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, \log_{\lvert\underline{x}\rvert} \underline{z}]$ | |
| Ⓨ | $\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, 0]$ | |
| Ⓩ | $\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, \log_{\lvert\overline{x}\rvert} \underline{z}]$ | |

*Note:* $\mathbb{Z}_1 = \{n \in \mathbb{Z} \mid n \text{ odd}\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z} \mid n \text{ even}\}$.

Table B.4 *(continued)*

| $\boldsymbol{z}\infty[-1,1]$ | $\boldsymbol{x}\infty[-1,0]$ finishes/overlappedBy/metBy | … |
|---|---|---|
| Ⓐ/Ⓑ | $\begin{cases} \mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, 0[ & \text{if } \underline{x} < \overline{x} \\ \emptyset & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓒ | $\begin{cases} \mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, \log_{\lvert\underline{x}\rvert} \lvert\overline{z}\rvert] & \text{if } \underline{x} < \overline{x} \\ \emptyset & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓓ/Ⓔ | $\begin{cases} \mathbb{Z}_1 \cap ]0, \log_{\lvert\underline{x}\rvert} \lvert\overline{z}\rvert] & \text{if } \underline{x} < \overline{x} \\ \emptyset & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓕ/Ⓖ/Ⓗ/Ⓜ/Ⓟ/Ⓢ | $\mathbb{R}^+$ | |
| Ⓘ/Ⓙ | $\begin{cases} (\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup \mathbb{R}^+ & \text{if } \underline{x} < \overline{x} \\ \mathbb{R}^+ & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓚ | $\begin{cases} (\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup \mathbb{R}^+ \cup \{0\} & \text{if } \underline{x} < \overline{x} \\ \mathbb{R}^+ & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓛ | $\begin{cases} (\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, +\infty[) \cup \mathbb{R}^+ \cup (\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, +\infty[) & \text{if } \underline{x} < \overline{x} \\ \mathbb{R}^+ & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓝ/Ⓠ/Ⓣ | $\begin{cases} \mathbb{R}^+ \cup \{0\} & \text{if } \underline{x} < \overline{x} \\ \mathbb{R}^+ & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓞ/Ⓡ/Ⓤ | $\begin{cases} \mathbb{R}^+ \cup (\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, +\infty[) & \text{if } \underline{x} < \overline{x} \\ \mathbb{R}^+ & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓥ | $\begin{cases} \mathbb{Z}_2 \cap ]0, \log_{\lvert\underline{x}\rvert} \underline{z}] & \text{if } \underline{x} < \overline{x} \\ \emptyset & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓦ | $\begin{cases} \mathbb{Z}_2 \cap [0, \log_{\lvert\underline{x}\rvert} \underline{z}] & \text{if } \underline{x} < \overline{x} \\ \emptyset & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓧ | $\begin{cases} \mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, \log_{\lvert\underline{x}\rvert} \underline{z}] & \text{if } \underline{x} < \overline{x} \\ \emptyset & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓨ | $\begin{cases} \mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, 0] & \text{if } \underline{x} < \overline{x} \\ \emptyset & \text{if } \underline{x} = \overline{x} \end{cases}$ | |
| Ⓩ | $\begin{cases} \mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, 0[ & \text{if } \underline{x} < \overline{x} \\ \emptyset & \text{if } \underline{x} = \overline{x} \end{cases}$ | |

*Note:* $\mathbb{Z}_1 = \{n \in \mathbb{Z} \mid n \text{ odd}\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z} \mid n \text{ even}\}$. For $\boldsymbol{x}$ overlappedBy/metBy [-1,0] the result is a union with $\mathrm{pow1}^{-}_{2}(\boldsymbol{x},\boldsymbol{z})$, see Table B.2.

TABLE B.4 *(CONTINUED)*

| | $\boldsymbol{x}$⚭[-1, 0] | |
|---|---|---|
| $\boldsymbol{z}$⚭⦶[-1, 1] | equalP/startedBy | … |
| Ⓐ | $\mathbb{Z}_1 \cap ]\text{-}\infty, 0[$ | |
| Ⓑ/Ⓒ/Ⓓ | $\mathbb{Z}_1$ | |
| Ⓔ | $\mathbb{Z}_1 \cap ]0, +\infty[$ | |
| Ⓕ/Ⓖ/Ⓟ/Ⓢ | $\mathbb{R}^+$ | |
| Ⓗ/Ⓘ/Ⓙ/Ⓜ | $\mathbb{Z}_1 \cup \mathbb{R}^+$ | |
| Ⓚ/Ⓛ/Ⓝ/Ⓞ | $\mathbb{Z} \cup \mathbb{R}^+$ | |
| Ⓠ/Ⓡ/Ⓣ/Ⓤ | $\mathbb{R}^+ \cup \mathbb{Z}_2$ | |
| Ⓥ | $\mathbb{Z}_2 \cap ]0, +\infty[$ | |
| Ⓦ/Ⓧ/Ⓨ | $\mathbb{Z}_2$ | |
| Ⓩ | $\mathbb{Z}_2 \cap ]\text{-}\infty, 0[$ | |

*Note:* $\mathbb{Z}_1 = \{n \in \mathbb{Z} \mid n \text{ odd}\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z} \mid n \text{ even}\}$. For $\boldsymbol{x}$ startedBy [-1, 0] the result is a union with $\text{pow1}^-_2(\boldsymbol{x}, \boldsymbol{z})$, see Table B.2.

Table B.4 *(continued)*

| | $\boldsymbol{x}$⚭[-1, 0] |
|---|---|
| $\boldsymbol{z}$⚭○[-1, 1] | finishedBy/containsP |
| Ⓐ | $\mathbb{Z}_1 \setminus [0, \log_{\lvert\underline{x}\rvert} \lvert\overline{z}\rvert[$ |
| Ⓑ/Ⓒ/Ⓓ | $\mathbb{Z}_1$ |
| Ⓔ | $\mathbb{Z}_1 \setminus ]\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, 0]$ |
| Ⓕ | $(\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, 0[) \cup \mathbb{R}^+$ |
| Ⓖ | $\mathbb{R}^+$ |
| Ⓗ/Ⓘ | $\mathbb{Z}_1 \cup \mathbb{R}^+$ |
| Ⓙ/Ⓜ | $\mathbb{Z}_1 \cup \mathbb{R}^+ \cup (\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, 0[)$ |
| Ⓚ/Ⓛ/Ⓝ/Ⓞ | $\mathbb{Z} \cup \mathbb{R}^+$ |
| Ⓟ | $(\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, 0[) \cup \mathbb{R}^+ \cup (\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, 0[)$ |
| Ⓠ/Ⓡ | $(\mathbb{Z}_1 \cap [\log_{\lvert\underline{x}\rvert} \lvert\underline{z}\rvert, 0[) \cup \mathbb{R}^+ \cup \mathbb{Z}_2$ |
| Ⓢ | $\mathbb{R}^+ \cup (\mathbb{Z}_2 \cap [\log_{\lvert\underline{x}\rvert} \overline{z}, 0[)$ |
| Ⓣ/Ⓤ | $\mathbb{R}^+ \cup \mathbb{Z}_2$ |
| Ⓥ | $\mathbb{Z}_2 \setminus ]\log_{\lvert\underline{x}\rvert} \overline{z}, 0]$ |
| Ⓦ/Ⓧ/Ⓨ | $\mathbb{Z}_2$ |
| Ⓩ | $\mathbb{Z}_2 \setminus [0, \log_{\lvert\underline{x}\rvert} \underline{z}[$ |

*Note:* $\mathbb{Z}_1 = \{n \in \mathbb{Z} \mid n \text{ odd}\}$, $\mathbb{Z}_2 = \{n \in \mathbb{Z} \mid n \text{ even}\}$. For $\boldsymbol{x}$ containsP [-1, 0] the result is a union with $\text{pow1}_2^-(\boldsymbol{x}, \boldsymbol{z})$, see Table B.2.

# Appendix C

# INTLAB Implementation

A reference implementation of the interval functions developed in the present work is provided on compact disk as an appendix. INTLAB is a common interval arithmetic library for the computational software Matlab developed by Rump (1999). However, INTLAB does not follow the current draft of the IEEE interval arithmetic standard, so there are a few issues to consider: In INTLAB there is no representation of empty intervals. I have used intervals which have boundaries that are not-a-number (nan) for this purpose. In INTLAB these intervals act as invalid data and mostly behave like empty intervals. One exception should be noticed: The hull operator returns invalid data, if one invalid parameter should be united with another interval. Hence, a custom hull operator is needed, which treats invalid intervals like empty ones.

An overview of implemented functions and their usage is given in Table C.1. In order to use the provided files, they should be located at the subdirectory "intval/@intval" of INTLAB's root directory.

TABLE C.1. Implemented INTLAB functions.

| | | |
|---|---|---|
| $\varnothing$ (overlap) | | |
| | File | intval/@intval/overlap.m |
| | Usage | overlap($\boldsymbol{a}$, $\boldsymbol{b}$) |
| | Returns | 'beforeP', 'meets', 'overlaps', 'starts', 'containedByP', 'finishes', 'equalP', 'finishedBy', 'containsP', 'startedBy', 'overlappedBy', 'metBy', or 'afterP' depending on $\boldsymbol{a} \circledcirc \boldsymbol{b}$ |
| (overlap extended) | | |
| | File | intval/@intval/overlap_extended.m |
| | Usage | overlap_extended($\boldsymbol{a}$, $\boldsymbol{b}$) |
| | Returns | 'A' … 'Z' depending on $\boldsymbol{a} \circledcirc \boldsymbol{b}$ |
| $\text{pow1}_{\mathbb{D}}$ | | |
| | File | intval/@intval/pow1.m |
| | Usage | pow1($\boldsymbol{x}$, $\boldsymbol{y}$) |
| | Returns | an enclosure of pow1 on $\boldsymbol{x}$ and $\boldsymbol{y}$ |
| $\text{pow1}^{-}_{1\,\mathbb{D}}$ | | |
| | File | intval/@intval/pow1_rev1.m |
| | Usage | pow1_rev1($\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$), or alternatively pow1_rev1($\boldsymbol{y}$, $\boldsymbol{z}$) |
| | Returns | an enclosure of $\text{pow1}^{-}_{1} \cap \boldsymbol{x}$ on $\boldsymbol{y}$ and $\boldsymbol{z}$ ($\boldsymbol{x} = \mathbb{R}$ in shortened usage) |
| $\text{pow1}^{-}_{2\,\mathbb{D}}$ | | |
| | File | intval/@intval/pow1_rev2.m |
| | Usage | pow1_rev2($\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$), or alternatively pow1_rev2($\boldsymbol{x}$, $\boldsymbol{z}$) |
| | Returns | an enclosure of $\text{pow1}^{-}_{2} \cap \boldsymbol{y}$ on $\boldsymbol{x}$ and $\boldsymbol{z}$ ($\boldsymbol{y} = \mathbb{R}$ in shortened usage) |
| | … | |

TABLE C.1 *(CONTINUED)*

| | | |
|---|---|---|
| $\mathrm{pow2}_{\mathbb{D}}$ | | |
| | File | intval/@intval/pow2.m |
| | Usage | pow2($\boldsymbol{x}$, $\boldsymbol{y}$) |
| | Returns | an enclosure of pow2 on $\boldsymbol{x}$ and $\boldsymbol{y}$ |
| $\mathrm{pow2}^{-}_{1\,\mathbb{D}}$ | | |
| | File | intval/@intval/pow2_rev1.m |
| | Usage | pow2_rev1($\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$), or alternatively pow2_rev1($\boldsymbol{y}$, $\boldsymbol{z}$) |
| | Returns | an enclosure of $\mathrm{pow2}^{-}_{1} \cap \boldsymbol{x}$ on $\boldsymbol{y}$ and $\boldsymbol{z}$ ($\boldsymbol{x} = \mathbb{R}$ in shortened usage) |
| $\mathrm{pow2}^{-}_{2\,\mathbb{D}}$ | | |
| | File | intval/@intval/pow2_rev2.m |
| | Usage | pow2_rev2($\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$), or alternatively pow2_rev2($\boldsymbol{x}$, $\boldsymbol{z}$) |
| | Returns | an enclosure of $\mathrm{pow2}^{-}_{2} \cap \boldsymbol{y}$ on $\boldsymbol{x}$ and $\boldsymbol{z}$ ($\boldsymbol{y} = \mathbb{R}$ in shortened usage) |
| $\mathrm{pow3}_{\mathbb{D}}$ | | |
| | File | intval/@intval/pow3.m |
| | Usage | pow3($\boldsymbol{x}$, $\boldsymbol{y}$) |
| | Returns | an enclosure of pow3 on $\boldsymbol{x}$ and $\boldsymbol{y}$ |
| $\mathrm{pow3}^{-}_{1\,\mathbb{D}}$ | | |
| | File | intval/@intval/pow3_rev1.m |
| | Usage | pow3_rev1($\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$), or alternatively pow3_rev1($\boldsymbol{y}$, $\boldsymbol{z}$) |
| | Returns | an enclosure of $\mathrm{pow3}^{-}_{1} \cap \boldsymbol{x}$ on $\boldsymbol{y}$ and $\boldsymbol{z}$ ($\boldsymbol{x} = \mathbb{R}$ in shortened usage) |
| $\mathrm{pow3}^{-}_{2\,\mathbb{D}}$ | | |
| | File | intval/@intval/pow3_rev2.m |
| | Usage | pow3_rev2($\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$), or alternatively pow3_rev2($\boldsymbol{x}$, $\boldsymbol{z}$) |
| | Returns | an enclosure of $\mathrm{pow3}^{-}_{2} \cap \boldsymbol{y}$ on $\boldsymbol{x}$ and $\boldsymbol{z}$ ($\boldsymbol{y} = \mathbb{R}$ in shortened usage) |

# Nomenclature

The following notation and symbols are used in the present work, partially without further explanation. Therefore, a very short introduction shall be given here.

| | |
|---|---|
| $f$ | Function, usually a point-function |
| $\boldsymbol{f}$ | Interval mapping, usually an interval function |
| $\boldsymbol{f}_{\mathbb{F}}$ | $\mathbb{F}$-interval mapping, usually a $\mathbb{F}$-interval function |
| $f_1^-$ | First reverse operation for $f$ |
| $f_2^-$ | Second reverse operation for $f$ |
| $\boldsymbol{f}_1^-$ | First reverse interval operation for $f$ |
| $\boldsymbol{f}_2^-$ | Second reverse interval operation for $f$ |
| $\boldsymbol{f}_{1\,\mathbb{F}}^-$ | First reverse $\mathbb{F}$-interval operation for $f$ |
| $\boldsymbol{f}_{2\,\mathbb{F}}^-$ | Second reverse $\mathbb{F}$-interval operation for $f$ |
| $0^-$ | Limit of a negative zero sequence |
| $0^+$ | Limit of a positive zero sequence |
| $\vert\ \vert$ | Absolute value of a number |
| $\circ$ | Abstract binary operator |
| exp | Exponential function |
| ! | Factorial of a positive integer |
| $\lfloor\ \rfloor$ | Floor function |
| hull | Tightest interval enclosure |

| | |
|---|---|
| $\mathrm{hull}_{\mathbb{F}}$ | Tightest $\mathbb{F}$-interval enclosure |
| inf | Infimum, i.e., the greatest lower bound, of a set |
| $\log_2$ | Binary logarithm |
| log | Natural logarithm |
| $\cdot$ | Operator for multiplication of real numbers |
| $\bullet$ | Interval multiplication operator |
| ⚭ | Overlapping Relation |
| ⚭⚬ | Extended overlapping relation |
| $\wp$ | Power set of a given set |
| $\diamond$ | Abstract directed rounding operator |
| $\triangledown$ | Downward directed rounding to the next smaller floating-point number |
| $\vartriangle$ | Upward directed rounding to the next greater floating-point number |
| sup | Supremum, i.e., the lowest upper bound, of a set |
| $\mathbb{D}$ | Set of double-precision numbers |
| $\mathbb{F}$ | Abstract number format, usually a floating-point format |
| $\overline{\mathbb{IF}}$ | Set of real, closed intervals with boundaries in $\mathbb{F}$ |
| $\overline{\mathbb{IR}}$ | Set of real, closed intervals |
| $\mathbb{N}$ | Set of natural numbers 1, 2, 3, ... |
| $\mathbb{Q}$ | Set of rational numbers |
| $\mathbb{R}$ | Set of real numbers |
| $\overline{\mathbb{R}}$ | Affinely extended real number system that includes $-\infty$ and $+\infty$ |
| $\mathbb{R}^-$ | Set of negative real numbers |
| $\mathbb{R}^+$ | Set of positive real numbers |
| $\mathbb{Z}$ | Set of integers |

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# Abschließende Erklärung

Ich, Oliver Heimlich, erkläre hiermit, dass die vorliegende Diplomarbeit von mir selbständig verfasst wurde. Informationen, die aus veröffentlichten oder unveröffentlichten Arbeiten Dritter gewonnen wurden, sind im Text kenntlich gemacht und sämtliche verwandte Quellen sind in einem Literaturverzeichnis aufgeführt.

Würzburg, 31. März 2011

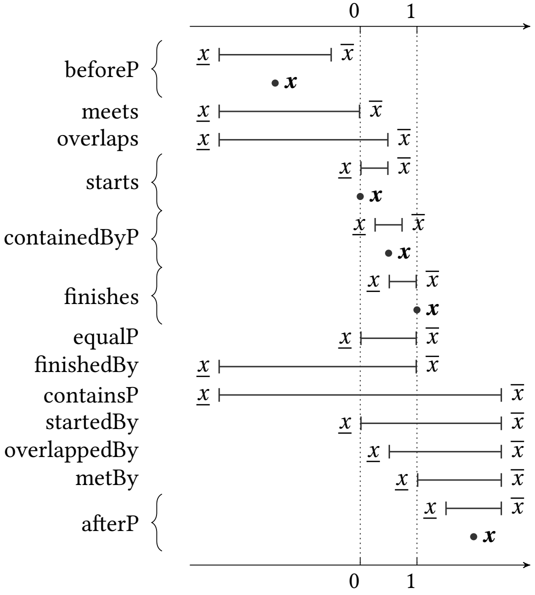


FIGURE. Overlapping states $\boldsymbol{x} \circledcirc [0,1]$ with non-empty, bound $\boldsymbol{x} = [\underline{x}, \overline{x}]$.

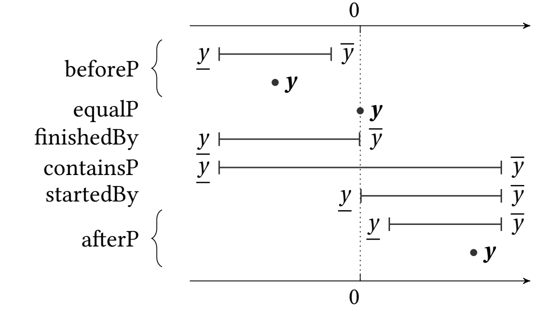


FIGURE. Overlapping states $\boldsymbol{y} \circledcirc [0,0]$ with non-empty, bound $\boldsymbol{y} = [\underline{y}, \overline{y}]$.

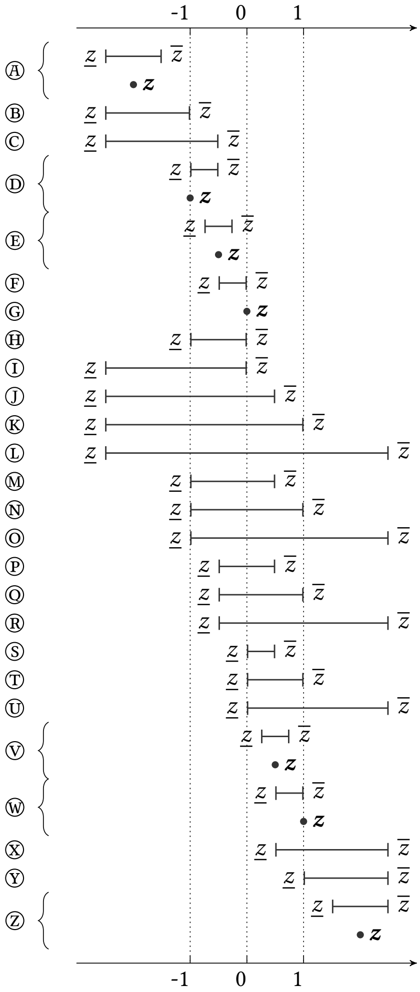


FIGURE. Overlapping states $\boldsymbol{z}$∞○$[-1, 1]$ with non-empty, bound $\boldsymbol{z} = [\underline{z}, \overline{z}]$.